ON GROUPS WHICH ARE PRODUCTS OF WEAKLY TOTALLY PERMUTABLE SUBGROUPS

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Abstract

This work is a contribution to the theory of products of finite groups. A group $G = AB$ is a weakly totally permutable product of subgroups A and B if every subgroup, U of A such that $U \leq A \cap B$ or $A \cap B \leq U$, permutes with every subgroup of B and if every subgroup V of B such that $V \leq A \cap B$ or $A \cap B \leq V$, permutes with every subgroup of A. It follows that a totally permutable product is a weakly totally permutable product. Some results on totally permutable products in the framework of formation theory are generalised. In particular it is shown that if the factors of a weakly totally permutable product are in $\mathcal{F}$, then the product is also in $\mathcal{F}$, where $\mathcal{F}$ is a formation containing $\mathcal{U}$, the class of all finite supersoluble groups. It is also shown that the $\mathcal{F}$-residual (and $\mathcal{F}$-projector) of the product G is just the product of the $\mathfrak{F}$-residuals (and respectively $\mathfrak{F}$-projectors) of the factors A and B, when $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$. Moreover, it is shown that a weakly totally permutable product is an SC-group if and only if its factors are SC-groups.

In the framework of Fitting classes some results are extended to weakly totally permutable products. Fischer classes containing $\mathfrak{U}$ were proved to behave nicely with respect to forming products in totally permutable products. It is shown that a particular Fischer class, $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$ and $\mathfrak{N}$ is the class of all nilpotent groups, also behave nicely with respect to forming products in weakly totally permutable products.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Dedication

Dedicated to my mother and the memory of my late father.

List of Notations

G, A, B	Sets and groups
g, h	Elements of a set
$\mathfrak{F}, \mathfrak{H}, \mathfrak{h}$	General classes of groups
$\mathfrak{S}, \mathfrak{U}, \mathfrak{N}$	Classes of all soluble, supersoluble and nilpotent groups respectively
$A \leq G$	A is a subgroup of G
$A < G$	A is a proper subgroup of G
$A \lhd G$	A is a normal subgroup of G
A sn G	A is a subnormal subgroup of G
AB	$\{ab \mid a \in A, b \in B\}$
G/N	Factor group of G with respect to the normal subgroup G
$\langle X \rangle$	Group generated by X
$\langle X, g \rangle$	$\langle X \cup \{g\} \rangle$
$\langle A^G \rangle$	Normal subgroup of G generated by A
$\|A\|$	Order of A
$\|G : A\|$	Index of A in G
$Z(G)$	Centre of G
$Z_{\mathfrak{F}}(G)$	$\mathfrak{F}$-hypercentre of G
$Z_{\infty}(G)$	Hypercentre of G
$G^{\mathfrak{F}}$	$\mathfrak{F}$-residual of G

$\Phi(G)$	Frattini subgroup of G
$F(G)$	Fitting subgroup of G
$G_{\mathfrak{F}}$	$\mathfrak{F}$-radical of G
$\mathrm{Core}_G(A)$	The core of A in G
$C_G(A)$	Centraliser of A in G
$N_G(A)$	Normaliser of A in G
$\mathrm{Aut}(G)$	Group of automorphisms of G
$A \times B$	Direct product of A with B
$[A]B$	Semi-direct product of A with B
$O^{\pi}(G)$	Group generated by all the p'-elements of G
$O_{\pi}(G)$	Product of all normal p-subgroups of G
a^b	$b^{-1}ab$
$[a,b]$	$a^{-1}b^{-1}ab$
$[A,B]$	Group generated by all $[a,b]$ with $a \in A$ and $b \in B$
$G' = [G,G]$	Derived subgroup of G
S_3	Symmetric group on 3 elements
$\mathbb{Z}_p$	The finite field of p-elements
$\mathfrak{F} \diamond \mathfrak{C}$	The Fitting product of $\mathfrak{F}$ with $\mathfrak{C}$
$\{N \cap A, N \cap B\} \subseteq \{N, 1\}$	Either $N \cap A = 1$ or $N \cap A = N$ and either $N \cap B = 1$ or $N \cap B = N$
(G,B)	The choice of a group G and its subgroup B
(G,A,B)	The choice of a group G and its subgroups A and B

Introduction

Unless otherwise stated, all groups in this thesis are finite.

In this thesis the structure of factorised groups is studied. Let a group $G = G_1G_2...G_n$ be a product of subgroups $G_1, G_2, ..., G_n$. We look at how the structure of G affects and is affected by the structure of the factors $G_1, G_2, ..., G_n$ when the factors have certain permutability properties. Asaad and Shaalan in [5] introduced two types of products of finite groups.

Definition 0.0.1. *Let a group $G = AB$ be the product of subgroups A and B. Then*
(i) G is a totally permutable product of subgroups A and B if every subgroup of A permutes with every subgroup of B.
(ii) G is a mutually permutable product of subgroups A and B if A permutes with every subgroup of B and B permutes with every subgroup of A.

A group $G = G_1G_2...G_n$ is a pairwise mutually (totally) permutable product of subgroups $G_1, G_2, ..., G_n$ if G_i (respectively every subgroup of G_i) permutes with every subgroup of G_j, for all $i, j \in \{1, 2, ..., n\}$, where $i \neq j$. The notion of a totally permutable product can be regarded as a generalisation of that of a central product while the notion of a mutually permutable product may be regarded as a generalisation of a normal product. It is known that the normal product of two supersoluble groups is not necessarily supersoluble. Asaad and Shaalan proved that a totally permutable product of two supersoluble groups is supersoluble. This contrast led many authors to study groups of this type and generate many results in the framework of formation theory, Fitting classes and other classes of finite groups [2, 3, 5-12, 16-20, 22, 23, 24, 25, 26, 29].
Peter Hauck defined a new type of product of finite groups:

Definition 0.0.2. *A group $G = AB$ is a weakly totally permutable product of subgroups A and B if every subgroup, U of A such that $U \leq A \cap B$ or $A \cap B \leq U$, permutes with every subgroup of B and if every subgroup V of B such that $V \leq A \cap B$ or $A \cap B \leq V$, permutes with every subgroup of A.*

Analogously a group $G = G_1G_2...G_n$ is the pairwise weakly totally permutable product of subgroups $G_1, G_2, ..., G_n$ if every subgroup, U of G_i such that $U \leq G_i \cap G_j$ or $G_i \cap G_j \leq U$, permutes with every subgroup of G_j, for all $i, j \in \{1, 2, ..., n\}$ where $i \neq j$.
It follows that a weakly totally permutable product is a mutually permutable product since $A \cap B \leq A$ and $A \cap B \leq B$. However the converse is not true as the following example shows.

Example 0.0.3. *Let $G = [T_7]S_3$ be the semi-direct product of T_7 and S_3 where T_7 is the non-abelian group of order 7^3 and exponent 7 whose presentation is*

$$T_7 = \langle a, b | a^7 = b^7 = [a, b]^7 = 1 \rangle$$

and S_3 is the symmetric group of degree 3, given by

$$S_3 = \langle x, y | x^3 = y^2 = 1, x^y = x^2 \rangle.$$

In this case S_3 acts on T_7 in the following way:

$$a^y = b, b^y = a, c^y = c^{-1}, a^x = a^2, b^x = b^4, c^x = c, \text{ where } c = [a, b].$$

Let $A = T_7\langle x \rangle$ and $B = G$. Then $G = AB$ is the mutually permutable product of subgroups A and B and $A \cap B = A$. Hence $\langle a \rangle \leq A \cap B = A$ and $\langle y \rangle \leq B = G$. But, $\langle a \rangle$ does not permute with $\langle y \rangle$. Hence $G = AB$ is not the weakly totally permutable product of subgroups A and B.

A totally permutable product is necessarily a weakly totally permutable product since all subgroups of A permute with all subgroups of B. However, the converse is not true as the following example shows.

Example 0.0.4. *Let*

$$G_1 = S_3 = \langle x, y | x^3 = y^2 = 1, x^y = x^2 \rangle$$

and

$$G_2 = \langle z | z^3 = 1 \rangle.$$

Let $G = G_1 \times G_2$. *Let* $A = \langle x, z \rangle$ *and* $B = \langle x, y \rangle$. *Since* A *has index* 2 *in* G, A *is a normal subgroup of* G. *Since* $B = G_1$ *is a direct factor of* G, B *is also normal in* G. *Therefore* $G = AB$ *is the product of normal subgroups* A *and* B. *Also,* $A \cap B = \langle x \rangle$ *is a simple normal subgroup of* G *since it is a normal subgroup of* G_1, *a direct factor of* G. *Subgroups of* G *that contain* $A \cap B$ *properly are either* A *or* B. *Hence* $G = AB$ *is the weakly totally permutable product of* A *and* B. *But ,* $\langle xz \rangle (\leq A)$ *does not permute with* $\langle y \rangle (\leq B)$. *Thus,* $G = AB$ *is not the totally permutable product of subgroups* A *and* B.

It is natural to ask which results on totally permutable products can be extended to weakly totally permutable products. The main objective of this thesis it to attempt to answer this question.

In Chapter 1 the fundamental concepts of group theory which are relevant in the following chapters are presented. Results on formation theory and some representation theory are also presented.

In Chapter 2 some results on totally permutable products in the framework of formation theory, most of which have proofs, are presented. The proofs are based on the ideas in [10, 11, 12, 22, 29].

In Chapter 3 some results on totally permutable products presented in Chapter 2 are extended to weakly totally permutable products. The main aim of Chapter 4 is presenting results on totally permutable products in the framework of Fitting classes and extending some of them to weakly totally permutable products. The thesis is brought to a close by looking at ideas to generalise some of the results on totally permutable products which were not generalised in the thesis.

Chapter 1

Fundamental Concepts

All groups considered in this thesis are finite unless otherwise stated.
In this chapter basic concepts of group theory are presented. In particular results on some special subgroups of a group such as Hall subgroups, Fitting subgroups and Frattini subgroups are presented. Results on formations are also presented. These results presented are for easy reference in the chapters to follow and they are presented without proofs. Most results in this chapter are taken from [23] and other texts such as [32, 27, 15, 4]. Notation in [23] is followed unless otherwise stated.

Let G be a group, A and B be subgroups of G. Then A permutes with B if $AB = BA$. The product AB is a subgroup of G if and only if A permutes with B. The subgroup A is a permutable subgroup of G if it permutes with every subgroup of G.
A fundamental result on subgroups of a group called the Dedekind Identity is useful:

Theorem 1.0.5. *(Dedekind Identity) Let A, B and C be subgroups of a group G with $B \leq A$. Then $A \cap BC = B(A \cap C)$.*

1.1 Hall Subgroups.

Let G be a group and p be a prime number that divides the order of G. The group G is a p-group if $|G| = p^n$ for some positive integer n. A subgroup of G which is a p-group is called a p-subgroup of G. A maximal p-subgroup of G is called a Sylow p-subgroup of G. Sylow p-subgroups for each prime p of a group always exist and any two Sylow p-subgroups are conjugate.

Let π be a non-empty set of primes and let π' be the set of all primes not in π. A positive integer n is a π-number if each prime divisor of n belongs to π. A group G is a π-group if the order of G is a π-number. A Sylow π-subgroup is a maximal π-subgroup of G. Sylow π-subgroups always exist but are not conjugate in general if π contains more than one prime.

A Hall π-subgroup of G is a π-subgroup H of G such that $|G : H|$ is a π'-number. It follows that Hall π-subgroups are Sylow π-subgroups. In general Hall π-subgroups do not always exist in a group.

We present a result known as the Frattini argument.

Theorem 1.1.1. *If H is a normal subgroup of G and P is a Sylow p-subgroup of H, then $G = N_G(P)H$.*

Below is an important result on subgroups whose subgroups are normalised by another subgroup.

Lemma 1.1.2. *[21, Corollary to Theorem 2.2.1] Let G be a group and let H and N be subgroups of G. If every subgroup of N is normalised by H, then $[N, H] \leq Z(N)$.*

1.2 Soluble, Supersoluble and Nilpotent Groups

Let G be a group and consider the series for G

$$H = G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G. \tag{1.1}$$

such that $G_i \lhd G_{i+1}$ for all $i \in \{1, 2, ..., n-1\}$. Then the series is called a subnormal series from H to G and H is a subnormal subgroup of G. If $G_i \lhd G$ for all $i \in \{1, 2, ..., n-1\}$, then the series (1.1) is a normal series. A chief series is a normal series such that G_{i+1}/G_i is a minimal normal subgroup of G/G_i for all $i \in \{1, 2, ..., n-1\}$. In this case the factors G_{i+1}/G_i are called chief factors of G. Two chief series of a finite group have the same number of terms and the chief factors are pairwise isomorphic with respect to a suitable ordering.

Definition 1.2.1. *A group G is soluble if it has an abelian series, that is, a subnormal series*

$$1 = G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G.$$

such that G_{i+1}/G_i is abelian for all $i \in \{1, 2, ..., n-1\}$.

Hall [23, I, Theorem 3.3(a)] proved that Hall π-subgroups always exist and are conjugate in a soluble group. Moreover he characterised soluble groups in terms of Hall subgroups:

Theorem 1.2.2. *A group G is soluble if and only if it possesses Hall π-subgroups for each set π of primes.*

A Hall system can now be defined. Let $\mathbb{P}$ represent the set of all primes.

Definition 1.2.3. *Let G be a soluble group and let $\sigma(G)$ denote the set of primes dividing $|G|$. A Hall system of G is a set Σ of Hall subgroups of G which satisfies the following conditions:*
(i) For each $\pi \subseteq \mathbb{P}, \Sigma$ contains exactly one Hall π-subgroup.
(ii) If $A, B \in \Sigma$, then $AB = BA$.

Let G be a group and let $x, y \in G$. The commutator of x with y is $[x, y] = x^{-1}y^{-1}xy$. If A and B are subgroups of G, then

$$[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle.$$

Now $[A, B] = 1$ if and only if $B \leq C_G(A)$ and $A \leq C_G(B)$. Also $[A, B] \trianglelefteq \langle A, B \rangle$. The commutator subgroup or the derived subgroup of G is defined as $G' = [G, G]$. Now G/G' is an abelian group. The series

$$G = G^{(0)} \rhd G' \rhd G^{(2)} \rhd ... \rhd G^{(n)} = G^{(n+1)}$$

where $G^{(i+1)} = (G^{(i)})' = [G^{(i)}, G^{(i)}]$ for $i \geq 1$ is called the derived series of G. The derived series is a series of characteristic subgroups of G. The factor $G^{(i)}/G^{(i+1)}$ is abelian for all $i \geq 1$. The last subgroup of the derived series, that is, the subgroup $[G^{(n)}, G^{(n)}] = (G^{(n)})'$, is called the soluble residual of G. The soluble residual of a group G is the smallest normal subgroup of G such that G/N is soluble. So G is soluble if and only if the soluble residual is trivial. Also a soluble group has a subnormal series in which its factors are abelian for example the derived series.

A group G is perfect if $[G, G] = G' = G$. It follows that the soluble residual of a non-soluble group G is perfect. In fact, it is a maximal normal perfect subgroup of G.

Definition 1.2.4. *A group G is supersoluble if it has a cyclic series, that is, a normal series*

$$1 = G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G$$

such that G_{i+1}/G_i is cyclic for each $i \in \{1, 2, ..., n-1\}$.

The chief factors of a supersoluble group are of prime order. A supersoluble group is soluble but the converse is not true in general. The symmetric group of degree 4, S_4 is an example of a soluble group which is not supersoluble. Supersoluble groups have the property that each maximal subgroup has prime index in the group.

Theorem 1.2.5. *Consider a group G in which the index of each maximal subgroup is a prime number. If p is the largest prime dividing $|G|$, then the corresponding Sylow p-subgroup is a normal subgroup of G.*

Therefore this result holds for supersoluble groups.

Definition 1.2.6. *A group G is nilpotent if it has a central series, that is, a normal series*

$$1 = G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_n = G$$

such that $G_{i+1}/G_i \leq Z(G/G_i)$ for all $i \in \{1, 2, ..., n-1\}$.

A nilpotent group is supersoluble but the converse is not true. The symmetric group of degree 3, S_3 is supersoluble but not nilpotent.

All p-groups are nilpotent. A direct product of nilpotent groups is also nilpotent. Several characterisations of nilpotent groups are given by the following theorem:

Theorem 1.2.7. *Let G be a group. Then the following properties are equivalent:*

(i) G is nilpotent.

(ii) Every subgroup of G is subnormal.

(iii) Every maximal subgroup of G is normal.

(iv) G is the direct product of its Sylow subgroups.

Lemma 1.2.8. *If G is a nilpotent group and $1 \neq N \lhd G$, then $N \cap Z(G) \neq 1$.*

The result below will be useful in Chapter 4.

Lemma 1.2.9. *[7, Lemma 1.3.4] Let p be a prime. If G is a non-abelian p-group, then the group of power automorphisms of G is an abelian p-group.*

1.3 The Fitting Subgroup, $O_p(G)$ and $O^p(G)$.

Fitting proved that the product of normal nilpotent subgroups is nilpotent. The Fitting subgroup of a group G, denoted by $F(G)$ is the subgroup generated by all normal nilpotent subgroups of G. This is the unique largest normal nilpotent subgroup of G, hence it is characteristic.

Theorem 1.3.1. *If G is a group, then $F(G)$ centralizes every chief factor of G.*

Hence $F(G)$ centralizes all minimal normal subgroups of G.

Definition 1.3.2. *Let G be a group.*

(i) The subgroup $O_p(G)$ for each prime p dividing $|G|$ is defined as:

$$O_p(G) = \langle N \mid N \trianglelefteq G \text{ and } N \text{ is a } p\text{-group } \rangle.$$

(ii) The subgroup $O^p(G)$ for each prime p dividing $|G|$ is defined by:

$$O^p(G) = \bigcap\{N \mid N \trianglelefteq G \textit{ and } G/N \textit{ is a } p\textit{-group }\}.$$

The subgroup $O^p(G)$ is generated by all Sylow q-subgroups for all primes $q \neq p$, dividing $|G|$. For each prime p subgroup $O_p(G)$ is the intersection of all Sylow p-subgroups of the group G. So the Fitting subgroup satisfies:

$$F(G) = \langle O_p(G) \mid p \text{ is a prime dividing } |G|\rangle.$$

Lemma 1.3.3. *For a soluble group $G \neq 1$, $F(G)$ is non-trivial.*

1.4 The Frattini Subgroup.

Let G be a group. The Frattini subgroup, denoted by $\Phi(G)$ is defined to be 1 when $G = 1$, otherwise $\Phi(G)$ is the intersection of all maximal subgroups of G. The Frattini subgroup is a characteristic subgroup and it is also nilpotent. Let G be a group and $g \in G$. Then g is a non-generator of G if for each subset X of G, $G = \langle g, X\rangle$ implies that $G = \langle X\rangle$. The Frattini subgroup is the set of all non-generators. Nilpotent, supersoluble and soluble groups can be characterised in terms of their Frattini subgroups.

Theorem 1.4.1. *Let G be a group. Then*

(i) G is nilpotent if and only if $G/\Phi(G)$ is nilpotent.

(ii) G is supersoluble if and only if $G/\Phi(G)$ is supersoluble.

(iii) G is soluble if and only if $G/\Phi(G)$ is soluble.

The following is another characterisation of nilpotent groups in terms of their Frattini subgroups.

Theorem 1.4.2. *Let G be a group. Then G is nilpotent if and only if $G' \leq \Phi(G)$.*

1.5 Primitive Groups.

Let G be a group and let H be a subgroup of G. The core of H in G, $\mathrm{Core}_G(H)$, is defined as the largest normal subgroup of G contained in H. If $\mathrm{Core}_G(H) = 1$, then H is core free.

Definition 1.5.1. *A group G is called primitive if it has a maximal subgroup M such that $Core_G(M) = 1$.*

In this case M is a stabiliser of G.

Definition 1.5.2. *Let G be a finite group and U and V be subgroups of G. Then U is supplemented by V in G if $UV = G$. If U is supplemented by V in G and $U \cap V = 1$, then U is complemented by V in G.*

Theorem 1.5.3. *Let G be a primitive group with stabiliser M. Then exactly one of the following three statements holds:*

(i) G has a unique minimal normal subgroup N, $N = C_G(N)$ and N is complemented by M in G.

(ii) G has a unique minimal normal subgroup N, N is non-abelian and N is supplemented by M in G.

(iii) G has exactly two minimal normal subgroups N_1 and N_2, and each of them is complemented by M in G. Also $C_G(N_1) = N_2$, $C_G(N_2) = N_1$ and $N_1 \cong N_2 \cong N_1N_2 \cap M$. Moreover, if $V < G$ and $VN_1 = VN_2 = G$, then $V \cap N_1 = V \cap N_2 = 1$.

So primitive groups fall into three categories and all the soluble primitive groups satisfy property (i). A group with property (i) is called a group of type 1.

Theorem 1.5.4. *Let G be a group.*

(a) If G is a primitive soluble group with stabiliser M, then

(i) G has a unique minimal normal subgroup N, for which $M \cap N = 1$, $MN = G$ and $N = C_G(N) = F(G)$.

(ii) If p is the prime dividing $|N|$, then $O_p(M) = 1$.

(b) G is a primitive group of type 1 if and only if (i) G has a unique minimal normal subgroup N, and (ii) N is abelian, and (iii) $N \not\leq \Phi(G)$.

A group is characteristically simple if it has no proper non-trivial characteristic subgroups. A finite group is characteristically simple if and only if it is a direct product of isomorphic simple groups. Minimal normal subgroups are characteristically simple. Let G be a group and N be a minimal normal subgroup of G. If N is abelian, then N is a direct product of isomorphic groups of order p for some prime p, that is, N is an elementary abelian p-group. If N is non-abelian, then N is a direct product of isomorphic non-abelian simple groups.

Theorem 1.5.5. *An elementary abelian p-group G of order p^n is isomorphic to a vector space of dimension n over the field $\mathbb{Z}_p$ with p elements.*

Hence abelian minimal normal subgroups can be viewed as vector spaces which will be useful in Chapter 2.

1.6 Formations.

A class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \cong G$, then $H \in \mathfrak{X}$.

Definition 1.6.1. *Let $\mathfrak{X}$ be a class of groups. Then*
(i) $\mathrm{Q}\mathfrak{X}$ is a class of groups with the following property:

$$\text{If } H \in \mathfrak{X} \text{ and there exists an epimorphism from } H \text{ onto } G, \text{ then } G \in \mathrm{Q}\mathfrak{X}.$$

(ii) $\mathrm{R}_0\mathfrak{X}$ is a class of groups with the following property:

$$\text{If } N_i \trianglelefteq G, \text{ for each } i = \{1, 2, ..., r\} \text{ with } G/N_i \in \mathfrak{X} \text{ and } \textstyle\bigcap_{i=1}^{r} N_i = 1, \text{ then } G \in \mathrm{R}_0\mathfrak{X}.$$

(iii) $\mathrm{S}_n\mathfrak{X}$ is a class of groups with the following property:

$$\text{If } G \text{ sn } H \text{ and } H \in \mathfrak{X}, \text{ then } G \in \mathrm{S}_n\mathfrak{X}.$$

(iv) $\mathrm{N}_0\mathfrak{X}$ is a class of groups with the following property:

$$\text{If } K_i \text{ sn } G, \text{ for each } i = \{1, 2, ..., r\} \text{ with } K_i \in \mathfrak{X}, \text{ then } G = \langle K_1, ..., K_r \rangle \in \mathrm{N}_0\mathfrak{X}.$$

Let G be a group. Then (G) denotes the smallest class containing the group G. The class of all primitive groups is denoted by $\mathfrak{B}$.

Definition 1.6.2. *A formation is a class $\mathfrak{F}$ of groups satisfying the following conditions:*
(a) If $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$.
(b) If $N_1, N_2 \trianglelefteq G$ with $N_1 \cap N_2 = 1$ and $G/N_i \in \mathfrak{F}$ for $i = 1, 2$, then $G \in \mathfrak{F}$.

The classes of all finite soluble, supersoluble and nilpotent groups are examples of formations and will be denoted by $\mathfrak{S}$, $\mathfrak{U}$ and $\mathfrak{N}$ respectively. The class $\mathfrak{S}_p$ denotes the class of all finite p-groups and is a formation. A formation is Q and R_0-closed, that is, if $\mathfrak{F}$ is a formation, then $Q\mathfrak{F} \subseteq \mathfrak{F}$ and $R_0\mathfrak{F} \subseteq \mathfrak{F}$.

Definition 1.6.3. *Let $\mathfrak{F}$ be a formation and let G be a group. The $\mathfrak{F}$-residual, $G^{\mathfrak{F}}$, of G is the smallest normal subgroup of G such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$.*

Theorem 1.6.4. *[23, IV, Theorem 1.18] Let $\mathfrak{F}$ be a formation of soluble groups and let $D = G_1 \times G_2 \times ... \times G_n$. Then $D^{\mathfrak{F}} = G_1^{\mathfrak{F}} \times G_2^{\mathfrak{F}} \times ... \times G_n^{\mathfrak{F}}$.*

A formation $\mathfrak{F}$ is saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. The classes $\mathfrak{S}$, $\mathfrak{U}$ and $\mathfrak{N}$ are examples of saturated formations.

Definition 1.6.5. *Let $\mathfrak{X}$ be a class of groups.*
(a) A maximal subgroup M of a group G is $\mathfrak{X}$-normal if

$$G/Core_G(M) \in \mathfrak{X}$$

otherwise it is said to be $\mathfrak{X}$-abnormal.
(b) A subgroup H of G is $\mathfrak{X}$-subnormal in G if either $H = G$ or there exist a chain

$$H = H_n \leq H_{n-1} \leq ... \leq H_0 = G$$

such that H_{i+1} is a maximal $\mathfrak{X}$-normal subgroup of H_i, for every $i \in \{0, 1, ..., n-1\}$.

Theorem 1.6.6. *[13, Theorem 3.7] Let $\mathfrak{F}$ be a subgroup-closed formation. Let G be a soluble group and H an $\mathfrak{F}$-subnormal subgroup of G. Suppose there exists a normal subgroup M such that $M \leq H \cap \Phi(G)$ and $H/M \in \mathfrak{F}$. Then $H \in \mathfrak{F}$.*

A homomorph $\mathfrak{X}$ is a class of groups such that if $G \in \mathfrak{X}$ and $N \lhd G$, then $G/N \in \mathfrak{X}$. Hence a formation is a homomorph.

Let $h : \mathbb{P} \longrightarrow \{group\ classes\}$ be a function which associates with each prime p a class(possibly empty) of groups $h(p)$. Let $\mathfrak{X}$ denote the class of all finite groups G which satisfy the following condition:

For all non-Frattini chief factors H/K of G and for all primes p dividing $|H/K|$, implies that

$$G/C_G(H/K) \in h(p).$$

The class $\mathfrak{X}$ is locally defined by h and is denoted by LC(h), that is, $\mathfrak{X} = \mathrm{LC}(h)$. The function h is a local function if $h(p)$ is a homomorph for all $p \in \mathbb{P}$. A class $\mathfrak{X}$ is a local class if $\mathfrak{X} = \mathrm{LC}(h)$ for some local function h.

Definition 1.6.7. *Let h be a local function and let $\mathfrak{h} = LC(h)$. Then h is*

(i) integrated if $h(p) \subseteq \mathfrak{h}$ for all $p \in \mathbb{P}$

(ii) full if $h(p) = \mathfrak{S}_p h(p)$ for all $p \in \mathbb{P}$.

Definition 1.6.8. *(a) A local function $f : \mathbb{P} \longrightarrow \{homomorphs\}$ is called a formation function if $f(p)$ is a formation for all $p \in \mathbb{P}$.*

(b) A class $\mathfrak{F}$ of finite groups is called a local formation if there exists a formation function f such that $\mathfrak{F} = LC(f)$. In this case we say $\mathfrak{F} = LF(f)$.

(c) If $f : \mathbb{P} \longrightarrow \{classes\ \ of\ \ groups\}$, a chief factor H/K of a group G is called f-central if

$$G/C_G(H/K) \in f(p) \text{ for all primes } p \text{ dividing } |H/K|.$$

Otherwise it is called f-eccentric.

So a group belongs to the class LC(f) if and only if its non-Frattini chief factors (it is sufficient for the chief factors to be from one chief series) are f-central.

However for local formations the restriction to non-Frattini chief factors is not necessary.

Theorem 1.6.9. *[23, IV, Theorem 3.2] Let f be a formation function. Then $G \in LF(f)$ if and only if all chief factors of G are f-central.*

In [23, IV, Theorem 3.7] it is shown that a formation function f which defines a local class $\mathfrak{F}$ that is full and integrated is unique.

Definition 1.6.10. *The uniquely determined full and integrated formation function defining a local formation $\mathfrak{F}$ is called the canonical local definition of $\mathfrak{F}$.*

Theorem 1.6.11. *(Gaschütz-Lubeseder-Schmid)[23, IV, Theorem 4.6]*
A formation of finite groups is saturated if and only if it is local.

The class $\mathfrak{U}$, of all finite supersoluble groups, is the local formation LF(u) defined by:

$$u(p) = \text{the formation of abelian groups of exponent dividing } p-1 \text{ for all primes } p.$$

Since $u(p) \subseteq \mathfrak{U}$, u is integrated. By [23, IV, Example 3.4(f)] supersoluble groups are characterised by the condition that their chief factors have prime order.

1.7 The $\mathfrak{F}$-hypercentre.

Let A be a set. A group G is called an A-group if there is associated with each element $a \in A$ an endomorphism of G denoted by $g \to ga$ for all $g \in G$.

Definition 1.7.1. *Let f be a formation function and G be an A-group. Then*
(a) A acts f-centrally on an A-composition factor H/K of G if $A/C_A(H/K) \in f(p)$ for all primes p dividing $|H/K|$, otherwise A acts f-eccentrically.
(b) A acts f-hypercentrally(respectively f-hypereccentrically) on G if it acts f-centrally(respectively f-eccentrically) on every A-composition factor of G.

Lemma 1.7.2. *[23, IV, Lemma 6.4(c)] Let f be a formation function, let G be an A-group and let M and N be A-invariant normal subgroups of G. If A acts f-hypercentrally on M and N, then it acts similarly on MN.*

Let f_1 and f_2 be two integrated local definitions of a local formation $\mathfrak{F}$. Then $Z_{f_1}(G) = Z_{f_2}(G)$ and hence we have the following definition.

Definition 1.7.3. *Let $\mathfrak{F} = LF(f)$ and let G be an A-group. Then a group possesses a unique maximal f-hypercentral normal subgroup denoted by $Z_f(G, A)$. When A is the group G acting by conjugation the unique maximal f-hypercentral normal subgroup is called the f-hypercentre of G, denoted by $Z_f(G)$. If f is integrated, then (since if f_1 and f_2 are integrated, $Z_{f_1}(G) = Z_{f_2}(G)$ and we have a unique f-hypercentre and) it is denoted by $Z_{\mathfrak{F}}(G)$ and called the $\mathfrak{F}$-hypercentre of G.*

Theorem 1.7.4. *[23, IV, Theorem 6.9] Let $\mathfrak{F} = LF(f)$ with f integrated and let G be an A-group such that $C_A(G) = 1$. If $Z_f(G, A) = G$, then $A \in \mathfrak{F}$.*

Corollary 1.7.5. *Let $\mathfrak{F} = LF(f)$ with f integrated and let G be an A-group. If $Z_f(G, A) = G$, then $A/C_A(G) \in \mathfrak{F}$.*

If $\mathfrak{F} = \mathfrak{N}$, the class of finite nilpotent groups, then $Z_{\mathfrak{N}}(G) = Z_\infty(G)$ is called the hypercentre of G.

There is a fundamental result by Maier and Schmid [30, Theorem] about permutable subgroups:

Theorem 1.7.6. *[30, Theorem] If H is a permutable subgroup of a finite group G, then $H^G/Core_G(H)$ is contained in the hypercentre $Z_\infty(G/Core_G(H))$ of $G/Core_G(H)$.*

So if H is core-free in the theorem above, then H is in the hypercentre $Z_\infty(G)$ of G.

Theorem 1.7.7. *[23, IV, Theorem 6.15] Let $\mathrm{S}_n\mathfrak{F} = \mathfrak{F} = LF(F)$, and let G be a group. Then $Z_{\mathfrak{F}}(G) \in \mathfrak{F}$.*

1.8 The $\mathfrak{F}$-projector.

Definition 1.8.1. *Let $\mathfrak{F}$ be a class of groups. A subgroup U of a group G is called an $\mathfrak{F}$-projector if UK/K is $\mathfrak{F}$-maximal in G/K for all $K \trianglelefteq G$.*

Denote the set of all $\mathfrak{X}$-projectors of G by $Proj_{\mathfrak{X}}(G)$, where $\mathfrak{X}$ is a class of groups.

Definition 1.8.2. *Let $\mathfrak{B}$ be the class of primitive groups. A subclass $\mathfrak{H}$ of $\mathfrak{B}$ is called $\mathfrak{B}$-projective if $Proj_{\mathfrak{H}}(G) \neq \varnothing$ for all $G \in \mathfrak{B}$.*

Definition 1.8.3. *A Schunck class $\mathfrak{X}$ is a class of finite groups if it satisfies the following conditions:*

(i) If $N \lhd G$ and $G \in \mathfrak{X}$, then $G/N \in \mathfrak{X}$.
(ii) If $N \lhd G$, $G/N \in \mathfrak{X}$ and G/N is a primitive group, then $G \in \mathfrak{X}$.

Saturated formations are Schunck classes.

Definition 1.8.4. *A $\mathfrak{B}$-Schunck class is a Schunck class contained in $\mathfrak{B}$.*

As mentioned in [23], Förster proved that $\mathfrak{B}$-projective classes can be characterised in terms of $\mathfrak{B}$-Schunck classes.

Theorem 1.8.5. *[23, III, Theorem 3.10] A class $\mathfrak{h}$ is $\mathfrak{B}$-projective if and only if it is a $\mathfrak{B}$-Schunck class.*

Hence saturated formations are $\mathfrak{B}$-projective. Results on projectors are presented below.

Theorem 1.8.6. *[23, III, Proposition 3.7] Let $\mathfrak{F}$ be a homomorph. If $N \trianglelefteq G$, such that $N \leq V \leq G$, U is an $\mathfrak{F}$-projector of V and V/N is an $\mathfrak{F}$-projector of G/N, then U is an $\mathfrak{F}$-projector of G.*

The next result shows a sufficient condition for an $\mathfrak{F}$-maximal of a group G to be an $\mathfrak{F}$-projector of G.

Lemma 1.8.7. *[23, III, Lemmas 3.14 and 3.18] Let $\mathfrak{F}$ be a saturated formation. Let $G = HN$ where H is an $\mathfrak{F}$-maximal subgroup of G. If N is either nilpotent or a direct product of non-abelian simple groups and N is normal in G, then H is an $\mathfrak{F}$-projector of G.*

Theorem 1.8.8. *[23, IV, Theorem 1.14] Let K be a nilpotent normal subgroup of a finite group G and let $G = WK$. Then $W \in \mathrm{QR}_0(G)$.*

Let $\mathfrak{F}$ be a saturated formation. The result below shows the relationship between an abelian $\mathfrak{F}$-residual and $\mathfrak{F}$-projectors of a group.

Theorem 1.8.9. *[23, IV, Theorem 5.18] Let $\mathfrak{F}$ be a saturated formation, let R denote the $\mathfrak{F}$-residual of a group G, and assume that R is abelian. Then R is complemented in G and each subgroup which complements R in G is an $\mathfrak{F}$-projector of G.*

Theorem 1.8.10. *[23, IV, Theorem 6.14] Let $\mathfrak{F}$ be a local formation and G be a group. If U is an $\mathfrak{F}$-maximal subgroup of G such that $G = UG^{\mathfrak{F}}$ (in particular if U is an $\mathfrak{F}$-projector of G), then*

$$Z_{\mathfrak{F}}(G) = C_U(G^{\mathfrak{F}}).$$

1.9 Representation of Groups

Definition 1.9.1. *Let R be a ring. A left R-module is an abelian group M together with a map $(a, m) \rightarrowtail am$ of $R \times M$ into M satisfying the following properties:*
(i) $a(m + n) = am + an$
(ii) $(a + b)m = am + bm$
(iii) $(ab)m = a(bm)$
(iv) $1m = m$
for $m, n \in M$ and $a, b \in R$. The map $R \times M \longrightarrow M$ is referred to as scalar multiplication.

The notion of a right R-module is dual to that of a left R-module. When R is commutative for example a field, there is no distinction between left and right modules. Every abelian group is a $\mathbb{Z}$-module.

Let V be a vector space over a field F of dimension n. The general linear group of dimension n is the group of $n \times n$ invertible matrices with coefficients in F, together with the operation of matrix multiplication. It is denoted by $GL(n, F)$. The general linear group can also be regarded as the set of all bijective linear transformations of V with composition of functions as the group operation, denoted $GL(V)$. If a fixed basis for V is chosen, each linear transformation of V is associated with an $n \times n$ matrix over F, and $GL(n, F) \cong GL(V)$.
A scalar matrix is a diagonal matrix which is a constant times the identity matrix. Scalar matrices are in the centre of $GL(n, F)$. Hence a scalar transformation is a linear transformation

corresponding to a scalar matrix with respect to the isomorphism.

Definition 1.9.2. *Let V be a vector space over a field F. A homomorphism ϕ of a group G into the group $GL(V)$ is called a representation of G. The vector space V is called a representation module for G.*

Kerϕ is called the kernel of the representation. If ϕ is injective, that is, ker$\phi = 1$, then ϕ is a faithful representation.

Theorem 1.9.3. *[27, Theorem 2.6.1] If the elementary abelian p-group H is regarded as a vector space over $\mathbb{Z}_p$, then Aut (H) is isomorphic to the group $GL(H)$.*

If H is normal, then there is a homomorphism from G into $GL(H)$. So H becomes a representation module for G and hence for subgroups of G.

Let ϕ be a faithful representation of G on a vector space V over a field F. If K is a subgroup of G inducing scalar transformations on V, then it is in the centre of G.
The fundamental results in this thesis have been presented in this chapter. In the next chapter results on totally and mutually permutable products of finite groups will be presented.

Chapter 2

Totally and Mutually Permutable Products of Finite Groups

The concept of a formation was presented in Chapter 1. In this chapter results on totally permutable and mutually permutable products in the framework of formations are presented. These results, which is a theory developed for over 20 years, are taken from [2-14, 16-20, 22, 24-25, 29, 33]. For many results on totally permutable products which do not hold for mutually permutable products, proofs are given. The results and proofs presented in this chapter will help in generating new ideas in Chapter 3 where we attempt to extend results on totally permutable products to weakly totally permutable products. These results will also be used in Chapter 4 where results on products of finite groups in the framework of Fitting classes are presented.

The first results are fundamental results on products of finite groups.

Theorem 2.0.4. *[4, Theorem 7.5.7] Let the finite group $G = AB$ be the product of two subgroups A and B. If H is a subgroup of $A \cap B$ which is subnormal in both A and B, then H is subnormal in G.*

Lemma 2.0.5. *[29, Lemma 1] Let $\langle x\rangle\langle y\rangle$ be the product of cyclic groups $\langle x\rangle$ and $\langle y\rangle$ of a group G. If $|G| \neq 1$ and $|\langle x\rangle| \geq |\langle y\rangle|$, then $\langle x\rangle$ contains a non-trivial normal subgroup of G. Moreover, G is supersoluble.*

Lemma 2.0.6. *[12, Lemma 1] Let the group $G = NB$ be the product of subgroups N and B. Suppose that N is normal in G. Since B acts by conjugation on N, the semidirect product*

$X = [N]B$ can be constructed, with respect to this action. Then the natural map $\alpha : X \to G$ given by $(nb)\alpha = nb$, for every $n \in N$ and every $b \in B$, is an epimorphism, Ker $\alpha \cap N = 1$ and Ker $\alpha \leq C_X(N)$.

2.1 Totally Permutable Products

In this section the result that the supersoluble residual of a factor centralises the other factor when G is a totally permutable product of two subgroups is presented. The first result below shows that factor groups of totally permutable products are also totally permutable products.

Lemma 2.1.1. *Let a group $G = AB$ be the totally permutable product of subgroups A and B. If N is a normal subgroup of G, then G/N is the totally permutable product of subgroups AN/N and BN/N.*

Proof. Let XN/N be a subgroup of AN/N and YN/N be a subgroup of BN/N. Then there exist subgroups $H \leq A$ and $K \leq B$ such that $HN = X$ and $KN = Y$. Since H permutes with K, it follows that HN/N permutes with $KN/N = Y/N$. Hence the result follows. □

R. Maier in [29] proved the following two lemmas.

Lemma 2.1.2. *[29, Lemma 2(a)] Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then the product $Core_G(A)Core_G(B) \neq 1$.*

Lemma 2.1.3. *[29, Lemma 2(b)] Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then the subgroup $A \cap B$ is a nilpotent subnormal subgroup of G.*

Proof. Let H be a subgroup of $A \cap B$. By definition H is a permutable subgroup and hence a subnormal subgroup in both A and B. By Theorem 2.0.4, H is a subnormal subgroup of $AB = G$. So H is a subnormal subgroup of $A \cap B$. It follows from Theorems 1.2.7 and 2.0.4 that $A \cap B$ is a nilpotent subnormal subgroup of G. □

Lemma 2.1.4. *[12, Lemma 2] Let a group $G = NB$ be the totally permutable product of subgroups N and B. Suppose N is a minimal normal subgroup of G.*
(a) If N is abelian, then N is a cyclic group of prime order.
(b) If N is non-abelian, then B centralizes N.

Proof. (a) Assume that N is abelian. Then N is a p-group for some prime p. Let P be a Sylow p-subgroup of B. Then NP is a p-group and N is a normal subgroup of NP. So $N \cap Z(NP) \neq 1$ by Lemma 1.2.8. Let $g \in N \cap Z(NP)$ be an element of order p. Then $\langle g \rangle$ is normal subgroup of P. Consider a Sylow q-subgroup Q of B, for some prime $q \neq p$. Then $Q\langle g \rangle$ is a subgroup and since $\langle g \rangle$ is subnormal in G, it is a subnormal Sylow p-subgroup of $Q\langle g \rangle$. So $\langle g \rangle$ is normalized by $Q\langle g \rangle$. Hence $\langle g \rangle$ is a normal subgroup of G and $N = \langle g \rangle$ is a cyclic group of prime order.
(b) Suppose that N is non-abelian. By Lemma 2.1.3, $N \cap B$ is a nilpotent subnormal subgroup of G. In particular, $N \cap B \leq F(N) = 1$. Let $X \leq N$. Then $X = X(N \cap B) = N \cap BX \trianglelefteq BX$. So B normalizes every subgroup of N. By Lemma 1.1.2, $[N, B] \leq Z(N) = 1$. Hence B centralizes N. □

Lemma 2.1.5. *[12, Lemma 3] Let the group $G = AB$ be the totally permutable product of subgroups A and B. If $A \cap B = 1$, then $[A, B]$ is contained in $F(G)$.*

Proof. By Theorem 1.3.1 it is sufficient to prove that $[A, B]$ centralizes all chief factors of a chief series. The proof is by induction on $|G|$. By Lemma 2.1.2 either A or B contains a non-trivial minimal normal subgroup of G. Assume that there exists a minimal normal subgroup N of G such that $N \leq A$. Consider a chief series of G that passes through N. By Lemma 2.1.1 G/N is the totally permutable product of subgroups AN/N and BN/N. So $[A, B]N/N$ centralizes all chief factors H/K of G such that $N \leq K$.
What is left is to show that $[A, B]$ centralizes N. Suppose N is non-abelian. Then N is direct product of isomorphic non-abelian simple groups $N_1 \times N_2 \times ... \times N_n$. Since these factors N_i are characteristic subgroups of N, N_i is a minimal normal subgroup of NB. Since BN_i is a totally permutable product of B and N_i, B centralizes N_i by Lemma 2.1.4. Hence B centralizes N. Since N is normal in G, N centralizes $\langle B^G \rangle = B[A, B]$. Therefore $[A, B]$ centralizes N.
Suppose N is abelian. Then N is an elementary abelian p-group for some p. So N can be viewed as a $G/C_G(N)$-module over the field with p elements. Note that $N \cap B \leq A \cap B = 1$. Let $X \leq N$.

Then $X = X(N \cap B) = N \cap BX \trianglelefteq BX$. This implies B normalizes every subgroup of N. So linear transformations induced by $BC_G(N)/C_G(N)$ must be scalar. Therefore $BC_G(N)/C_GN$ centralizes $AC_G(N)/C_G(N)$ and so $[A, B] \leq C_G(N)$. Hence the result follows. $\square$

Lemma 2.1.6. *[12, Lemma 7] Let the group $G = AB$ be the totally permutable product of subgroups A and B. Assume $A \cap B = 1$ and $F(G) = O_p(G)$ for some prime p. Then :*
(a) $[O^p(A), O^p(B)] = 1$, and
(b) $O^p(A)$ normalizes each p-subgroup of B.

Proof. (a) Let q and r be two primes such that $p \neq q$ and $p \neq r$. Let Q be a Sylow q-subgroup of A and let R be a Sylow r-subgroup of B. Since QR is a totally permutable product of subgroups Q and R, it follows that $[Q, R] \leq [A, B] \leq F(G)$ by Lemma 2.1.5. Since $F(G) = O_p(G)$, $[Q, R]$ is a p-group. But QR is a subgroup of G so $[Q, R] \leq QR$ which is a π-group where $\pi = \{q, r\}$. So $[Q, R] = 1$. Since $O^p(A) = \langle Q \mid Q$ is a Sylow q-subgroup of A, $q \neq p\rangle$ it follows that $[O^p(A), O^p(B)] = 1$ as required.
(b) Let P be a p-subgroup of B and let Q be a Sylow q-subgroup of A, where q is a prime dividing $|A|$ and $q \neq p$. Then QP is the totally permutable product of Q and P. So QP is a subgroup of G. Since $[Q, P] \leq [A, B] \leq F(G) = O_p(G)$, $[Q, P]$ is a p-subgroup and it is also normal in QP. So $[Q, P] \leq P$ since P is a Sylow p-subgroup of QP. Hence Q normalizes P. So $[Q, R] = 1$. Since $O^p(A) = \langle Q \mid Q$ is a Sylow q-subgroup of A, $q \neq p\rangle$, the result follows. $\square$

The following result proved by Maier [29] extends Asaad and Shaalan's result in [5].

Theorem 2.1.7. *[29, Theorem] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the totally permutable product of A and B. If A and B belong to $\mathfrak{F}$, then G also belongs to $\mathfrak{F}$.*

Proof. Suppose the theorem is not true and let G be a minimal counterexample. Then G satisfies the following conditions:

(i) There exists a unique abelian minimal normal subgroup N of G and $\Phi(G) = 1$, that is, G is a primitive group of type 1 and $F(G) = C_G(N) = N$.

Suppose there exists two minimal normal subgroups N_1 and N_2 of G. Then G/N_1 and G/N_2 belong to $\mathfrak{F}$ and hence $G/(N_1 \cap N_2) \cong G \in \mathfrak{F}$ which is a contradiction. Hence G has a unique minimal normal subgroup N. Since $\mathfrak{F}$ is a saturated formation, it is clear that $\Phi(G) = 1$.

By Lemma 2.1.2 and the uniqueness of N, assume that $N \leq A$. Suppose that N is non-abelian. Then N is the direct product of non-abelian simple groups. Since N is unique, $F(G) = 1 = C_G(N)$. By Lemma 2.1.3, $A \cap B \leq F(G) = 1$. So $A \cap B = 1$. Since $N \leq A$, $N \cap B = 1$ and B permutes with every subgroup of N. If $X \leq N$, then $X = XB \cap N \trianglelefteq XB$, so B normalizes X. By Lemma 1.1.2, $[N, B] \leq Z(N) = 1$, that is, $1 \neq B \leq C_G(N) = 1$, a contradiction. Hence N is abelian.

Therefore by Theorem 1.5.4 G is a primitive group of type 1. Now $O_q(G) = 1$ for all primes $q \neq p$ and $O_p(G) \leq F(G) \leq C_G(N) = N$. Hence $F(G) = C_G(N) = N$. Let M be the stabiliser of G. So $MN = G$ and $M \cap N = 1$.

(ii) $|N| > p$, N is complemented by $V = A \cap M$ in A and $A \cap B = 1$.

If $|N| = p$, then $G/C_G(N)$ is abelian of exponent $p - 1$. So $G/C_G(N) \in U(p) \subseteq F(p)$, where $U(p)$ and $F(p)$ are canonical local definitions for $\mathfrak{U}$ and $\mathfrak{F}$. Hence $G \in \mathfrak{U} \subseteq \mathfrak{F}$, a contradiction. Therefore $|N| > p$.

Now $V \cap N = (A \cap M) \cap N \leq M \cap N = 1$. Also $VN = (A \cap M)N = A \cap G = A$. Hence N is complemented by V in A. Suppose $A \cap B \neq 1$. Then by Lemma 2.1.3 $A \cap B \leq F(G) = N$. So $A \cap B = N \cap B \trianglelefteq B$. Now $(A \cap B)V = V(A \cap B)$ if $A \cap B$ is regarded as a subgroup of B. So $A \cap B = N \cap (A \cap B)V \trianglelefteq (A \cap B)V$. Since N is abelian $A \cap B$ is a normal subgroup of A. Hence $A \cap B = N$. Then $N \leq B$ and N is complemented by $W = M \cap B$. Let $H \leq N$. Then $H = VH \cap N \trianglelefteq VH$ and $H = WH \cap N \trianglelefteq WH$. Hence $H \trianglelefteq (VH)N = G$. Therefore N must be cyclic of order p which contradicts the first part of (ii). Hence the result follows.

(iii) B normalizes all subgroups of N and B centralizes V^a, $a \in A$.

Since $N \cap B = 1$, it follows that $Y = Y(N \cap B) = N \cap BY \trianglelefteq BY$ for all $Y \leq N$. Hence B normalizes every subgroup of N. Also $[A, B] \leq F(G) \leq N$ by Lemma 2.1.5. Let $a \in A$. So $[B, V^a] \leq N$. But BV^a is a subgroup hence $[B, V^a] \leq BV^a$. Therefore $[B, V^a] \leq BV^a \cap N = 1$

and the result follows.

(iv) V is a maximal subgroup of A and N is a minimal normal subgroup of A.

Let $X \leq N$ be a normal subgroup of A. Since B normalizes X by (iii), X is a normal subgroup of G. Hence $X = N$ and so N is a minimal normal subgroup of A. Since $A = NV$ and N is an abelian minimal normal subgroup of A, it follows that V is a maximal subgroup of A.

(v) Final contradiction.

Since B centralizes V^a and $A = \langle V^a \mid a \in A \rangle$, it follows that B centralizes A. By (ii), $G = A \times B$ and hence $G \in \mathfrak{F}$, our final contradiction. □

The converse of Theorem 2.1.7 was proved by Ballester-Bolinches and Pérez-Ramos in [12].

Theorem 2.1.8. *[12, Lemma 4] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the totally permutable product of subgroups A and B. If G belongs to $\mathfrak{F}$, then A and B belong to $\mathfrak{F}$.*

Proof. The proof is by induction on $|G|$. If G has two minimal normal subgroups N_1 and N_2, then $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$. So both $AN_1/N_1 \cong A/(A \cap N_1)$ and $AN_2/N_2 \cong A/(A \cap N_2)$ belong to $\mathfrak{F}$ which implies that $A/(A \cap N_1 \cap N_2) \cong A$ belong to $\mathfrak{F}$. The same applies for B. Hence assume that G has a unique minimal normal subgroup N and the subgroups AN/N and BN/N are in $\mathfrak{F}$. So either $N \subseteq \Phi(G)$ or $N \nsubseteq \Phi(G)$.
(i) $N \subseteq \Phi(G)$.
Then N is abelian. Since $AN/N \cong A/(A \cap N) \in \mathfrak{F}$, it follows that $A^{\mathfrak{F}} \leq A \cap N$ is abelian and by Theorem 1.8.9, $A = (A \cap N)F$, where F is an $\mathfrak{F}$-projector of A. So $G = (A \cap N)FB = FB$ and $A \cap G = A \cap FB = F(A \cap B)$. Now $A = F(A \cap B)$ is a totally permutable product of subgroups F and $A \cap B$ and $A \cap B \leq F(G) \in \mathfrak{U} \subseteq \mathfrak{F}$ by Lemma 2.1.3. Hence $A \in \mathfrak{F}$ by Theorem 2.1.7. The same applies for B.
(ii) $N \nsubseteq \Phi(G)$.
If N is non-abelian, then $A \cap B \leq F(G) = 1$. By Lemma 2.1.5, $[A, B] \leq F(G) = 1$. Hence

$G = A \times B$ and the result follows.

Assume N is abelian. Then by Theorem 1.5.4 G is a primitive group of type 1. Let M be the stabilizer of G. Then M is a maximal subgroup of G such that $G = NM$, $M \cap N = 1$ and $\mathrm{Core}_G(M) = 1$. By Lemma 2.1.2 assume that $N \leq A$. Let $U = M \cap A$. Then $A = NU$ and $N \cap U = 1$. Since $A \cap B \leq F(G) = N$ by Lemma 2.1.3 and Theorem 1.5.4, $A \cap B = N \cap B \trianglelefteq B$. If $T \leq N \cap B$, then $T = TU \cap N \trianglelefteq TU$ and $T = TK \cap N \trianglelefteq TB$. This implies that A and B normalize every subgroup of $A \cap B$. In particular either $A \cap B = 1$ or $A \cap B = N$. If $A \cap B = N$, then N is a group of order p for some prime p. Therefore $A/C_A(N) \in U(p) \subseteq F(p)$, where U and F are the canonical local definitions of $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Since $A/N \in \mathfrak{F}$, $A \in \mathfrak{F}$. With the same argument $B \in \mathfrak{F}$.

If $A \cap B = 1$, then $N \cap B = 1$. Let $K \leq N$. Then $K = KB \cap N \trianglelefteq KB$. Hence B normalizes every subgroup of N. This implies that N is a minimal normal subgroup of A. Since N is abelian, U is a maximal subgroup of A. If $x \in N$, then $[U^x, B] \leq F(G) \cap U^x B = 1$. If $N = A$, then A is abelian hence it belongs to $\mathfrak{F}$. If $N < A$, then since $H = \langle U, U^x \rangle$ it follows that $B \leq C_G(\langle U, U^x \rangle) = C_G(H)$. Therefore $G = A \times B$ and the result follows. □

Theorem 2.1.9. *[12, Lemma 5] Let a group $G = AB$ be the totally permutable product of subgroups A and B. If B is supersoluble, then $G^{\mathfrak{U}} = A^{\mathfrak{U}}$.*

Proof. Since $G/G^{\mathfrak{U}}$ is supersoluble, it follows that $AG^{\mathfrak{U}}/G^{\mathfrak{U}} \cong A/(A \cap G^{\mathfrak{U}})$ is supersoluble by Theorem 2.1.8 and so $A^{\mathfrak{U}} \leq G^{\mathfrak{U}}$. The proof is by induction on $|G|$. Let N be a minimal normal subgroup of G. Then G/N satisfies the hypothesis. So $(G/N)^{\mathfrak{U}} = G^{\mathfrak{U}}N/N = A^{\mathfrak{U}}N/N$. Hence $G^{\mathfrak{U}}N = A^{\mathfrak{U}}N$. Since $G^{\mathfrak{U}} = G^{\mathfrak{U}} \cap A^{\mathfrak{U}}N = A^{\mathfrak{U}}(G^{\mathfrak{U}} \cap N)$ it follows that $N \leq G^{\mathfrak{U}}$ for any minimal normal subgroup N of G and $G^{\mathfrak{U}} = A^{\mathfrak{U}}N$. By Lemma 2.1.2 there exists a minimal normal subgroup N of G such that $N \leq A$ or $N \leq B$. If $N \leq A^{\mathfrak{U}}$, then the result follows.

If $N \leq A$, then $G^{\mathfrak{U}} \leq A$ and $N \leq N_G(A^{\mathfrak{U}})$ since $A^{\mathfrak{U}}$ is normal in A. So $A^{\mathfrak{U}} \trianglelefteq G^{\mathfrak{U}}$. Suppose N is non-abelian. Then N is the direct product of non-abelian simple groups. But $G^{\mathfrak{U}}/A^{\mathfrak{U}} \leq A/A^{\mathfrak{U}}$ is supersoluble and $G^{\mathfrak{U}}/A^{\mathfrak{U}} = A^{\mathfrak{U}}N/A^{\mathfrak{U}} \cong N/(N \cap A^{\mathfrak{U}})$. So $N/N \cap A^{\mathfrak{U}}$ is supersoluble which implies that $N = N \cap A^{\mathfrak{U}}$. Hence $N \leq A^{\mathfrak{U}}$ and $G^{\mathfrak{U}} = NA^{\mathfrak{U}} = A^{\mathfrak{U}}$.

Suppose N is abelian. Then N is a p-group. Suppose there exists a Sylow q-subgroup Q of $G^{\mathfrak{U}}$ for some prime $q \neq p$. Then $Q \leq A^{\mathfrak{U}}$ and $G = G^{\mathfrak{U}}N_G(Q)$ by Theorem 1.1.1. Hence $1 \neq \langle Q^G \rangle = \langle Q^{G^{\mathfrak{U}}} \rangle \leq A^{\mathfrak{U}}$. This means there exists a minimal normal subgroup of G which is a subgroup of $A^{\mathfrak{U}}$ and the result follows. Assume that $G^{\mathfrak{U}}$ is a p-group. If $\Phi(G^{\mathfrak{U}}) \neq 1$, then $G^{\mathfrak{U}} = \Phi(G^{\mathfrak{U}})A^{\mathfrak{U}} = A^{\mathfrak{U}}$ since $\Phi(G^{\mathfrak{U}}) \trianglelefteq G$. Therefore $\Phi(G^{\mathfrak{U}}) = 1$ and by Theorem 1.4.2 $G^{\mathfrak{U}}$ is abelian. So G is soluble. Consider $A = A^{\mathfrak{U}}U$, where U is an $\mathfrak{U}$-projector of A by Theorem 1.8.5. Then $G = AB = A^{\mathfrak{U}}(UB) = G^{\mathfrak{U}}(UB)$ since $A^{\mathfrak{U}} \leq G^{\mathfrak{U}}$. By Theorem 2.1.7 UB is supersoluble. Since $G^{\mathfrak{U}}$ is abelian, there exists an $\mathfrak{U}$-projector M of G such that $UB \leq M$ and $M \cap G^{\mathfrak{U}} = 1$ by Theorem 1.8.9. Hence $G^{\mathfrak{U}} = G^{\mathfrak{U}} \cap A^{\mathfrak{U}}UB = A^{\mathfrak{U}}(G^{\mathfrak{U}} \cap UB) = A^{\mathfrak{U}}$ and the result follows. Assume $N \leq B$. Since B is supersoluble, N is abelian. If $AN < G$, then by induction $(AN)^{\mathfrak{U}} = A^{\mathfrak{U}}$. In particular N normalizes $A^{\mathfrak{U}}$ and so $A^{\mathfrak{U}} \trianglelefteq G^{\mathfrak{U}}$. The result follows arguing as in the case where $N \leq A$. Assume that $AN = G$. Then G is the totally permutable product of subgroups N and A and A is a maximal subgroup of G since N is abelian. By Lemma 2.1.4 N is a cyclic group of order p for some prime p. Hence $A/C_A(N)$ is supersoluble which implies $A^{\mathfrak{U}} \leq C_A(N)$. If $Core_G(A) \neq 1$, then there exists a minimal normal subgroup of A and the result follows by arguing as in the case where $N \leq A$. If $Core_G(A) = 1$, then $C_A(N) = 1$. Moreover since $N \not\leq \Phi(G)$, G is a primitive group of type 1. Hence $A^{\mathfrak{U}} \leq C_A(N) = 1$ and A is supersoluble. Therefore G is supersoluble by Theorem 2.1.7 and the result follows. □

Lemma 2.1.10. *[12, Lemma 6] Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then $A^{\mathfrak{U}}$ and $B^{\mathfrak{U}}$ are normal subgroups of G.*

Proof. Let q be a prime dividing $|G|$ and let Q be a Sylow q-subgroup of B. Since A permutes with every subgroup of B, AQ is a subgroup of G. By Lemma 2.1.9 $(AQ)^{\mathfrak{U}} = Q^{\mathfrak{U}}$ since Q is nilpotent hence supersoluble. This implies that $A^{\mathfrak{U}}$ is a normal subgroup AQ for every Sylow subgroup Q of B. Hence $A^{\mathfrak{U}}$ is a normal subgroup of $AB = G$. Analogously $B^{\mathfrak{U}}$ is a normal subgroup of G. □

Lemma 2.1.11. *[12, Lemma 8] Let a group $G = AB$ be the totally permutable product of subgroups A and B. If B is supersoluble, then B centralizes $G^{\mathfrak{U}}$.*

Proof. By Theorem 2.1.9, $G^{\mathfrak{U}} = A^{\mathfrak{U}}$. Suppose the lemma is not true and let (G, B) be a counterexample with $|G| + |B|$ minimal. Then G satisfies the following properties:

(i) B is a cyclic group of prime power order.

Let M be a maximal subgroup of B. Then AM is the totally permutable product of subgroups A and M and $(AM)^{\mathfrak{U}} = A^{\mathfrak{U}}$ by Lemma 2.1.9. Since $|MA| + |M| < |G| + |B|$ it follows that $[A^{\mathfrak{U}}, M] = 1$. If M_1 and M_2 are two maximal subgroups of B, then $B = \langle M_1, M_2 \rangle \leq C_G(A^{\mathfrak{U}})$ which is a contradiction. Therefore B has a unique maximal subgroup. This means that B is a cyclic group of prime power order.

(ii) G has a unique minimal normal subgroup N and $N \leq A^{\mathfrak{U}} = G^{\mathfrak{U}}$.

Let N be a minimal normal subgroup of G. Then G/N is the totally permutable product of subgroups AN/N and BN/N and BN/N is supersoluble. Since $|G/N| < |G|$ it follows that $BN/N \leq C_{G/N}(G^{\mathfrak{U}}N/N)$. So if N_1 and N_2 are two minimal normal subgroups of G, then $[B, G^{\mathfrak{U}}] \leq N_1 \cap N_2 = 1$, which is a contradiction. Hence G has a unique minimal normal subgroup N. Since $A^{\mathfrak{U}} = G^{\mathfrak{U}}$ is a normal subgroup of G, $N \leq A^{\mathfrak{U}}$.

(iii) B is a cyclic group of prime order.

Let M be the unique maximal subgroup of B. Assume that $M \neq 1$. Since $|AM| + |M| < |G| + |B|$, it follows that $M \leq C_G(A^{\mathfrak{U}})$. Let H be an $\mathfrak{U}$-projector of A by Theorem 1.8.5. Then $A = HA^{\mathfrak{U}}$. Since HB is the totally permutable product of subgroups H and B, HB is supersoluble by Theorem 2.1.7. Also $G = AB = A^{\mathfrak{U}}(HB)$. Let U be a $\mathfrak{U}$-maximal subgroup of G such that $HB \leq U$. Then $G = UA^{\mathfrak{U}}$. From Theorem 1.8.10, $Z_{\mathfrak{U}}(G) = C_U(G^{\mathfrak{U}})$. Hence $Z_{\mathfrak{U}}(G)$ is a non trivial normal subgroup of G since $1 \neq M \leq Z_{\mathfrak{U}}(G)$. Since N is unique in G it follows that $N \leq Z_{\mathfrak{U}}(G)$ which means that $Z_{\mathfrak{U}}(G/N) = Z_{\mathfrak{U}}(G)/N$. Since $G = UA^{\mathfrak{U}}$, $G/N = (A^{\mathfrak{U}}/N)(UN/N)$ and $(G/N)^{\mathfrak{U}} = A^{\mathfrak{U}}/N$ by Lemma 2.1.9. Let U_1/N be a $\mathfrak{U}$-maximal subgroup of G/N containing UN/N. By Theorem 1.8.10, $Z_{\mathfrak{U}}(G)/N = Z_{\mathfrak{U}}(G/N) = C_{U_1/N}(A^{\mathfrak{U}}/N)$. Since $|G/N| < |G|$ it follows that $BN/N \leq Z_{\mathfrak{U}}(G)/N$ and so $B \leq Z_{\mathfrak{U}}(G) = C_G(G^{\mathfrak{U}})$ which is a contradiction. Hence $M = 1$ and therefore B is a cyclic group of prime order.

(iv) Final contradiction.

Let $|B| = p$ for some prime p. Suppose B is a subgroup of A. Then B is permutable in

A.

If $\mathrm{Core}_G(B) \neq 1$, then B is a normal subgroup of G. Since $|B| = p$ and N is unique in G, it follows that $B \leq Z_{\mathfrak{U}}(G) \leq C_G(A^{\mathfrak{U}})$, a contradiction.

Suppose $\mathrm{Core}_G(B) = 1$. Since B is permutable in G, $B \leq Z_\infty(G)$ by Theorem 1.7.6. Hence $B \leq Z_\infty(G) \leq Z_{\mathfrak{U}}(G) \leq C_G(G^{\mathfrak{U}})$, is a contradiction.

Assume that B is not a subgroup of A. Then $A \cap B = 1$ and by Lemma 2.1.5 $[A, B] \leq F(G)$. If $[A, B] = 1$, then the result follows. So $1 \neq [A, B] \leq F(G)$. If q and r are two primes dividing $|F(G)|$, then $O_q(G) \neq 1$ and $O_r(G) \neq 1$ which contradicts the uniqueness of N. So $F(G)$ is a q-group for some prime q. If $q = p$, then $O^q(A) \leq N_G(B)$ by Lemma 2.1.6(b). Since $A/O^q(A)$ is a q-group, $A^{\mathfrak{U}} \leq O^q(A)$. It follows that $[B, A^{\mathfrak{U}}] \leq A$. Hence $[B, A^{\mathfrak{U}}] = A \cap B = 1$ which implies B centralizes $A^{\mathfrak{U}}$, a contradiction. Therefore $q \neq p$. By Lemma 2.1.6(i) and the fact that $O^q(B) = B$ it follows that $[O^q(A), O^q(B)] = [O^q(A), B] = 1$, that is, B centralizes $O^q(A)$. Since $A^{\mathfrak{U}} \leq O^q A$, B centralizes $A^{\mathfrak{U}}$, our final contradiction. □

The main result of this section can now be presented. This result was proved by Ballester-Bolinches *et al.* in [10].

Corollary 2.1.12. *[10, Corollary] Let $G = AB$ be the totally permutable product of subgroups A and B. Then $[A, B^{\mathfrak{U}}] = [A^{\mathfrak{U}}, B] = 1$.*

Proof. Suppose the corollary is not true and let G be a counterexample with $|G| + |B|$ minimal. Let M be a maximal subgroup of B. Then AM is the totally permutable product of subgroups A and M. By the choice of G, it follows that $M \leq C_G(A^{\mathfrak{U}})$. Let M_1 and M_2 be two distinct maximal subgroups of B. So $B = \langle M_1, M_2 \rangle \leq C_G(A^{\mathfrak{U}})$, a contradiction. Hence B has a unique maximal subgroup. This means that B is a cyclic group of prime power order. By Lemma 2.1.11 B centralizes $A^{\mathfrak{U}}$, our final contradiction. □

2.2 Pairwise Totally Permutable Products

In this section results on pairwise totally permutable products in the framework of formation theory are presented.

Lemma 2.2.1. *[22, Lemma 1] Let a group $G = G_1G_2...G_n$ be the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$.*
(a) Then there exists $i \in \{1, 2, ..., n\}$ such that G_i contains a non-trivial normal subgroup of G.
(b) For every $i, j \in \{1, 2, ..., n\}$, $i \neq j$, $G_i \cap G_j \leq F(G_iG_j)$.

Proof. (a) Let p be the largest prime dividing $|G|$. Then p divides one of $|G_1|, |G_2|, ..., |G_n|$. Let x be a p-element of the union set $G_1 \cup G_2 \cup ... \cup G_n$ of maximal order. Assume that $x \in G_1$. Since $\langle x \rangle$ is cyclic there exists a unique subgroup R of order p in $\langle x \rangle$. Note that R is a characteristic subgroup of every non-empty subgroup of $\langle x \rangle$. Let y be a q-element of G_i, where q is a prime and $i \in \{2, ..., n\}$. Since G_1 and G_2 are totally permutable the subgroup $\langle x \rangle \langle y \rangle$ is a supersoluble group by Lemma 2.0.5.
If $q \neq p$, then $\langle x \rangle$ is a Sylow p-subgroup of $\langle x \rangle \langle y \rangle$. Since $p > q$, $\langle x \rangle$ is a normal subgroup by Theorem 1.2.5. If $q = p$, then $|\langle x \rangle| \geq |\langle y \rangle|$ and by Lemma 2.0.5 there exists a non-trivial normal subgroup of $\langle x \rangle \langle y \rangle$ contained in $\langle x \rangle$. So R is a characteristic subgroup of that normal subgroup contained in $\langle x \rangle$. So in both cases, the unique subgroup R of order p in $\langle x \rangle$ is normalised by y. Since y is an arbitrary element of prime power order of G_i, G_i normalizes R for all $i \in \{2, ..., n\}$. Hence the normal closure

$$\langle R^G \rangle = \langle R^{G_1G_2...G_n} \rangle = \langle R^{G_2G_3...G_nG_n} \rangle = R^{G_1} \leq G_1$$

is a non-trivial normal subgroup of G contained in G_1. Hence the result follows.

(b) By Lemma 2.1.3 $G_i \cap G_j$ is a nilpotent subnormal subgroup of G_iG_j and hence the result follows. □

Lemma 2.2.1(a) extends Lemma 2.1.2 to any finite number of factors.

Lemma 2.2.2. *[11, Lemma 1] Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. Then $G_i^{\mathfrak{F}}$ is a normal subgroup of G for all $i \in \{1, 2, ..., n\}$.*

Proof. Since $G_i/G_i^{\mathfrak{U}} \in \mathfrak{U} \subseteq \mathfrak{F}$ it follows that $G_i^{\mathfrak{F}} \leq G_i^{\mathfrak{U}}$. By Corollary 2.1.12 $G_i^{\mathfrak{F}}$ centralizes $G_2G_3...G_n$. Since $G_i^{\mathfrak{F}}$ is a normal subgroup of G_i, $G_i^{\mathfrak{F}}$ is a normal subgroup of G. □

Lemma 2.2.3. *Let a group $G = G_1G_2...G_n$ be the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If G_i is supersoluble for all $i \in \{1, 2, ..., n\}$, then G is supersoluble.*

Proof. Suppose the lemma is not true and let G be a minimal counterexample. Since $\mathfrak{U}$ is a saturated formation it follows that G has a unique minimal normal subgroup N and $\Phi(G) = 1$. By Lemma 2.2.1 there exists $i \in \{1, 2, ..., n\}$ such that $N \leq G_i$. Assume that $N \leq G_1$. Since G_1 is supersoluble, N is an abelian p-group for some prime p. So G is soluble. By Theorem 1.5.4 G is a primitive group of type 1. Let M be the stabilizer of G. Then $O_p(M) = 1$ by Theorem 1.5.4. Hence N is the Sylow p-subgroup of G. Let $M \cap G_1 = H_1$. Then $N \cap H_1 = 1$ and $H_1N = G_1$ and H_1 is a p'-subgroup of G_1. Moreover for all $i \in \{2, 3, ..., n\}$ either $G_i = H_i$ or there exists a Hall p'-subgroup H_i of G_i. In particular $H_i \cap N = 1$ for all $i \in \{2, 3, ..., n\}$. Hence $H_1H_2...H_n$ is a Hall p'-subgroup of G and $N(H_1H_2...H_n) = G$. Suppose $N \cap G_i \neq 1$ for some $i \in \{2, 3, ..., n\}$. Assume that $N \cap G_2 \neq 1$. Let $X \leq N \cap G_2$ be a cyclic group of prime order. Then $X = X(H_1 \cap N) = XH_1 \cap N \trianglelefteq XH_1$. Moreover $X = XH_i \cap N \trianglelefteq XH_i$ for all $i \in \{2, 3, ..., n\}$. Hence $H_1H_2...H_n$ normalizes N. Since N is abelian, X is a normal subgroup of G and hence $N = X$ is a cyclic group of order p. Now $G/C_G(N)$ is abelian of order $p - 1$. This implies G is supersoluble, a contradiction. Therefore $N \cap G_i = 1$ for all $i \in \{2, 3, ..., n\}$. If $T \leq N$, then $T = G_iT \cap N \trianglelefteq G_iT$ for all $i \in \{2, 3, ..., n\}$. So $G_2G_3...G_n$ normalizes T. On the other hand $T = H_1T \cap N \trianglelefteq H_1T$. Since N is abelian T is a normal subgroup of G and N is cyclic of prime order. This implies G is supersoluble, which is our final contradiction. □

Lemma 2.2.3 is generalised to any formation containing $\mathfrak{U}$ in the following result.

Theorem 2.2.4. *[11, Theorem 1] Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If for all $i \in \{1, 2, ..., n\}$ the subgroups G_i are in $\mathfrak{F}$, then $G \in \mathfrak{F}$.*

Proof. Suppose the theorem is not true and let G be a counterexample with $|G|+|G_1|+|G_2|+ ...+|G_n|$ minimal. By Lemma 2.2.3 there exists $i \in \{1, 2, ..., n\}$ such that G_i does not belong to $\mathfrak{U}$. Assume that $G_1^{\mathfrak{U}} \neq 1$. By Lemma 2.1.11 $G_1^{\mathfrak{U}}$ centralizes $K = G_2G_3...G_n$. Since this implies that $G_1^{\mathfrak{U}}$ is a normal subgroup of G, it follows that $G_1^{\mathfrak{U}}$ centralizes $\langle K^G\rangle$. Moreover, $G_1 = G_1^{\mathfrak{U}}U$, where U is an $\mathfrak{U}$-projector of G_1 by Theorem 1.8.5. Consider UK. Now UK is the pairwise totally permutable product of subgroups $U, G_2, G_3, ..., G_n$. So
$|UK| + |U| + |G_2| + |G_3| + ... + |G_n| < |G| + |G_1| + |G_2| + ... + |G_n|$ and $UK \in \mathfrak{F}$. Since $\langle K^G\rangle$ is a normal subgroup of G, G_1 acts on $\langle K^G\rangle$ by conjugation. Let $Y = [\langle K^G\rangle]G_1$ be the semidirect product of $\langle K^G\rangle$ and G_1 with respect to this action. By Lemma 2.0.6 there is an epimorphism from Y onto G. So G is isomorphic to a factor group of Y. So it is sufficient to show that Y belongs to $\mathfrak{F}$.
Since $G_1^{\mathfrak{U}}$ centralizes $\langle K^G\rangle$ it follows that $\langle K^G\rangle = \langle K^U\rangle \leq UK$ which belongs to $\mathfrak{F}$. So $Y/G_1^{\mathfrak{U}} \cong [\langle K^G\rangle](U/(U \cap G_1^{\mathfrak{U}}))$, which is a factor group of $[\langle K^G\rangle]U$. By Lemma 2.0.6 there exists an epimorphism ψ from $[\langle K^U\rangle]U$ onto UK. So $[\langle K^U\rangle]U/\langle K^U\rangle$ is isomorphic to $U \in \mathfrak{U} \subseteq \mathfrak{F}$. Hence $[\langle K^U\rangle]U/(\text{Ker } \psi \cap U) \cong [\langle K^U\rangle]U \in \mathfrak{F}$ since $\langle K^U\rangle \cap \text{Ker } \psi = 1$ by Lemma 2.0.6. So $Y/G_1^{\mathfrak{U}} \in \mathfrak{F}$. On the other hand $Y/\langle K^G\rangle$ is isomorphic to $G_1 \in \mathfrak{F}$. Therefore $Y \cong Z/(\langle K^G\rangle \cap G_1^{\mathfrak{U}}) \in \mathfrak{F}$, a contradiction. Hence the result follows. □

The converse of Theorem 2.2.4 is not true in general (see [7, Example 5.2.6]). However the following result will help in showing that the converse of Theorem 2.2.4 is true if $\mathfrak{F}$, containing $\mathfrak{U}$, is either a saturated formation or a formation of soluble groups.

Lemma 2.2.5. *[11, Lemma 2] Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If $G_2, G_3, ..., G_n$ and G belong to $\mathfrak{F}$, then $G_1 \in \mathfrak{F}$.*

Proof. If G_1 is supersoluble, then the result follows. Assume that $G_1^{\mathfrak{U}} \neq 1$. So $G_1 = G_1^{\mathfrak{U}}U$, where U is an $\mathfrak{U}$-projector of G_1. By Lemma 2.2.2 $G_1^{\mathfrak{U}}$ is a normal subgroup of G_1. Hence U

acts on $G_1^{\mathfrak{U}}$ by conjugation. Let $Y = [G_1^{\mathfrak{U}}]U$ be the semidirect product of $G_1^{\mathfrak{U}}$ and U with respect to this action.

By Lemma 2.0.6 there is an epimorphism $\gamma : Y \longrightarrow G_1$. So G_1 is isomorphic to a quotient group of Y. By Lemma 2.2.2 $G_1^{\mathfrak{U}}$ is a normal subgroup of G. Let $K = G_2G_3...G_n$. Then $G = G_1^{\mathfrak{U}}(UK)$. Now UK is the pairwise totally permutable product of subgroups $U, G_2, G_3, ..., G_n$. Hence UK belongs to $\mathfrak{F}$ by Theorem 2.2.4. So UK acts on $G_1^{\mathfrak{U}}$ by conjugation. Let $X = [G_1^{\mathfrak{U}}]UK$ be the semidirect product with respect to this action. By Lemma 2.0.6 there exists an epimorphism $\alpha : X \longrightarrow G$ with $\text{Ker}\alpha \cap G_1^{\mathfrak{U}} = 1$. Hence $X/\text{Ker}\alpha \cong G$ belongs to $\mathfrak{F}$. On the other hand $X/G_1^{\mathfrak{U}} \cong UK \in \mathfrak{F}$. Hence $X/(\text{Ker}\alpha \cap G_1^{\mathfrak{U}}) \cong X \in \mathfrak{F}$.

By Corollary 2.1.12 $G_1^{\mathfrak{U}}$ centralizes K. It follows that $\langle K^X \rangle = \langle K^{\mathfrak{U}} \rangle \leq KU$. So $\langle K^X \rangle$ is contained in KU. Hence $\langle K^X \rangle \cap G_1^{\mathfrak{U}} = 1$. Moreover $X/\langle K^X \rangle \cong [G_1^{\mathfrak{U}}](U/(\langle K^X \rangle \cap U))$. Hence $[G_1^{\mathfrak{U}}]U/(\langle K^X \cap U) = Y$ belongs to $\mathfrak{F}$. Since $Y/G_1^{\mathfrak{U}} = [G_1^{\mathfrak{U}}]U/G_1^{\mathfrak{U}} \cong U$ is supersoluble,
$Y/(\langle K^X \rangle \cap G_1^{\mathfrak{U}} \cap U) \cong Y \in \mathfrak{U}$. Hence $Y \in \mathfrak{F}$ and so $G_1 \in \mathfrak{F}$. □

The next two results show that the converse of Theorem 2.2.4 holds when $\mathfrak{F}$ is either a saturated formation or a formation of soluble groups.

Theorem 2.2.6. *[11, Theorem 2] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If $G \in \mathfrak{F}$, then $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$.*

Proof. The proof is by induction on $|G|$. Assume that G has a unique minimal normal subgroup N. Since G/N satisfies the theorem, $G_iN/N \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$. By Lemma 2.2.1 there exists $i \in \{1, 2, ..., n\}$ such that $N \leq G_i$. Assume that $N \leq G_1$. Let $j \in \{1, 2, ..., n\}$ such that $j \neq 1$. Since $G_jN/N \cong G_j/(N \cap G_j) \in \mathfrak{F}$ it follows that $G_j = F_j(G_j \cap N)$, where F_j is an $\mathfrak{F}$-projector of G_j by Theorem 1.8.5. So $N \cap G_j \leq N \leq G_1$ and $F_j \leq G_j$. This means that $N \cap G_j \leq G_1 \cap G_j \leq F(G_1G_j)$ which is supersoluble. Hence $G_j = F_j(G_j \cap N)$ is the totally permutable product of subgroups $F_j \in \mathfrak{F}$ and $N \cap G_j \in \mathfrak{F}$. By Theorem 2.2.4 $G_j \in \mathfrak{F}$. Therefore $G_j \in \mathfrak{F}$ for all $j \in \{2, 3, ..., n\}$. The result follows by Lemma 2.2.5. □

Theorem 2.2.7. *[11, Theorem 3] Let $\mathfrak{F}$ be a formation of soluble groups containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If $G \in \mathfrak{F}$, then $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$.*

Proof. Suppose the theorem is not true and let G be a counterexample with $|G| + |G_1| + |G_2| + ... + |G_n|$ minimal. Assume that there exists $i \in \{1, 2, ..., n\}$ such that G_i does not belong to $\mathfrak{U}$. Without loss of generality assume that $G_1^{\mathfrak{U}} \neq 1$. If $G_1^{\mathfrak{U}} \leq \Phi(G_1)$, then $G \in \mathfrak{U} \subseteq \mathfrak{F}$, a contradiction. So $G_1^{\mathfrak{U}}$ is not contained in $\Phi(G_1)$. Since $G \in \mathfrak{F}$, it follows that G is soluble. In particular G_1 is soluble and hence $G_1^{\mathfrak{U}}/(G_1^{\mathfrak{U}} \cap \Phi(G_1))$ is soluble. So the Fitting subgroup $F/(G_1^{\mathfrak{U}} \cap \Phi(G_1))$ of $G_1^{\mathfrak{U}}/(G_1^{\mathfrak{U}} \cap \Phi(G_1))$ is non-trivial. By Theorem 1.6.6 F is nilpotent since $F/(G_1^{\mathfrak{U}} \cap \Phi(G_1))$ is nilpotent. Note that F is subnormal in G since $F \trianglelefteq G_1^{\mathfrak{U}} \trianglelefteq G$. This implies that $F \leq F(G)$. On the other hand F is a normal subgroup of G_1 which is not contained in $\Phi(G_1)$. This means that there exists a maximal subgroup M of G_1 such that F is not contained in M. Hence $G_1 = FM$ and $G = F(MG_2G_3...G_n) = F(G)(MG_2G_3...G_n)$. By Theorem 1.8.8, $MG_2G_3...G_n \in \mathfrak{F}$ since a formation is QR_0 closed. Consider $J = MG_2G_3...G_n$. Then J is the pairwise totally permutable product of $M, G_2, G_3, ..., G_n$. Also
$|J| + |M| + |G_2| + ... + |G_n| < |G| + |G_1| + |G_2| + ... + |G_n|$.
By the choice of G it follows that $G_2, G_3, ..., G_n$ belong to $\mathfrak{F}$. Using Lemma 2.2.5 it follows that $G_1 \in \mathfrak{F}$ and hence the result follows. $\square$

2.3 Totally permutable products and formation subgroups

The first three results of this section were proved by Ballester-Bolinches, Pérez-Ramos and Pedraza-Aguilera in [11].

Theorem 2.3.1. *[11, Theorem 4] Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$ such that either $\mathfrak{F}$ is saturated or $\mathfrak{F}$ is a formation of soluble groups. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. Then $G^{\mathfrak{F}} = G_1^{\mathfrak{F}}G_2^{\mathfrak{F}}...G_n^{\mathfrak{F}}$.*

Proof. The proof is by induction on $|G|$. If $G^{\mathfrak{F}} = 1$ the result follows from Theorems 2.2.7 and 2.2.6. So assume that $G^{\mathfrak{F}} \neq 1$. The factor group $G/G^{\mathfrak{F}}$ inherits the hypothesis of the theorem.

Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ it follows that $G_iG^{\mathfrak{F}}/G^{\mathfrak{F}} \cong G_i/(G_i \cap G^{\mathfrak{F}})$ belongs to $\mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$. This means that $G_i^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ for all $i \in \{1, 2, ..., n\}$. By Lemma 2.2.2 $G_i^{\mathfrak{F}}$ is a normal subgroup of G for all $i \in \{1, 2, ..., n\}$. So the product $H = G_1^{\mathfrak{F}}G_2^{\mathfrak{F}}...G_n^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ is a normal subgroup of G. Let $H = 1$, then $G^{\mathfrak{F}} = 1$ by Theorem 2.2.4. Assume that $H \neq 1$. Let N be a minimal normal subgroup of G contained in H. Since G/N satisfies the hypothesis of the theorem $G^{\mathfrak{F}}/N = (G_1^{\mathfrak{F}}N/N)(G_2^{\mathfrak{F}}N/N)...(G_n^{\mathfrak{F}}N/N)$ by induction. Hence $G^{\mathfrak{F}} = G_1^{\mathfrak{F}}G_2^{\mathfrak{F}}...G_n^{\mathfrak{F}}N = HN = H$ and the result follows. □

The result above generalises Theorems 2.2.6 and 2.2.7. The next two results show the relationship between the $\mathfrak{F}$-projectors of the factors and that of the product where the group is a totally permutable product.

Theorem 2.3.2. *[11, Theorem 5] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If A_i is an $\mathfrak{F}$-projector of G_i for all $i \in \{1, 2, ..., n\}$, then the product $A_1A_2...A_n$ is an $\mathfrak{F}$-projector of G.*

Proof. Suppose the theorem is not true and let G be a counterexample with $|G| + |G_1| + |G_2| + ... + |G_n|$ minimal. Assume $G_i \neq 1$ for all $i \in \{1, 2, ..., n\}$. By Lemma 2.2.1 assume that there exist a minimal normal subgroup N of G such that $N \leq G_1$. The factor group G/N satisfies the hypothesis of the theorem. By the choice of G it follows that $(A_1N/N)(A_2N/N)...(A_nN/N) = (A_1A_2...A_n)N/N$ is an $\mathfrak{F}$-projector of G/N.
Let $C = (A_1A_2...A_n)N$ be a proper subgroup of G. Now C is the pairwise totally permutable product of subgroups $A_1N, A_2, ..., A_n$. Suppose C is a proper subgroup of G. By Lemma 1.8.7 A_1 is an $\mathfrak{F}$-projector of A_1N. Since $|C| + |A_1N| + |A_2| + ... + |A_n| < |G| + |G_1| + |G_2| + ... + |G_n|$ it follows that $A_1A_2...A_n$ is an $\mathfrak{F}$-projector of C. By Theorem 1.8.6 $A_1A_2...A_n$ is an $\mathfrak{F}$-projector of G which is a contradiction.
Hence $G = C = (A_1A_2...A_n)N$. Now G/N is isomorphic to a factor group of $A_1N, A_2, ..., A_n$ which belongs to $\mathfrak{F}$ by Theorem 2.2.4. So $G^{\mathfrak{F}} \leq N$. Since $G^{\mathfrak{F}}$ is a normal subgroup of G and N is a minimal normal subgroup of G it follows that $G^{\mathfrak{F}} = 1$ or $G^{\mathfrak{F}} = N$. If $G^{\mathfrak{F}} = 1$, then $G \in \mathfrak{F}$ and $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$ by Theorem 2.2.6, a contradiction. Therefore $G^{\mathfrak{F}} = N$. Since $A_1A_2...A_n \in \mathfrak{F}$, there exists an $\mathfrak{F}$-maximal subgroup U of G such that $A_1A_2...A_n \leq U$.

But $G = (A_1A_2...A_n)N = UN$. Hence by Lemma 1.8.7 U is an $\mathfrak{F}$-projector of G. Also $U = (A_1N)A_2...A_n \cap U = (A_1(N \cap U))A_2...A_n$ is the pairwise totally permutable product of subgroups $A_1(U \cap N), A_2, ..., A_n$. Since $U, A_2, ..., A_n \in \mathfrak{F}$ it follows that $A_1(U \cap N) \in \mathfrak{F}$ by Lemma 2.2.5. Since A_1 is $\mathfrak{F}$-maximal in G_1, it also follows that $U \cap N \leq A_1$. Hence $U = A_1A_2...A_n$ is an $\mathfrak{F}$-projector of G, our final contradiction. □

Beidleman and Heineken [16] generalized Corollary 2.1.12 by showing that the nilpotent residual of one factor centralizes the other factor when G is a torsion group. The finite case was proved by Ballester-Bolinches *et al.* [7] using a different approach from that of Beidleman and Heineken.

Theorem 2.3.3. *[7, Theorem 4.2.7] Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then $[A^{\mathfrak{N}}, B] = [A, B^{\mathfrak{N}}] = 1$.*

Corollary 2.3.4. *[24, Lemma 3] Let a group $G = AB$ be a totally permutable product of subgroups A and B. Then $[A, B] \leq Z_{\mathfrak{U}}(G)$ and $A \cap B \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Theorem 2.3.1 $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$. Let A_1 and B_1 be $\mathfrak{U}$-projectors of A and B, respectively using Theorem 1.8.5. Then $A = A_1A^{\mathfrak{U}}$ and $B = B_1B^{\mathfrak{U}}$. Then $[A, B] = [A^{\mathfrak{U}}A_1, B^{\mathfrak{U}}B_1] = [A_1, B_1] \leq A_1B_1$. Moreover, A_1B_1 is a $\mathfrak{U}$-projector of G by Theorem 2.3.2. But $[A, B] \leq \langle A^G \rangle \cap \langle B^G \rangle \leq C_G(A^{\mathfrak{U}}B^{\mathfrak{U}}) = C_G(G^{\mathfrak{U}})$. It follows that $[A_1, B_1] \leq C_{(A_1B_1)}(G^{\mathfrak{U}}) = Z_{\mathfrak{U}}(G)$ by Theorem 1.8.10.
Now $(A \cap B)Z_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G) \leq Z(G/Z_{\mathfrak{U}}(G)) = 1$ which implies that $A \cap B \leq Z_{\mathfrak{U}}(G)$. □

The result below shows the generalisation of Corollary 2.3.4.

Lemma 2.3.5. *[25, Lemma 3] Let the group $G = G_1G_2...G_n$ be a pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. Then*

$$[\textstyle\prod_{i \in I} G_i, \prod_{j \in J} G_j] \leq Z_{\mathfrak{U}}(G)$$

for any $I, J \subseteq \{1, 2, ..., n\}$ such that $I \cap J = \emptyset$. In particular,

$$(\textstyle\prod_{i \in I} G_i) \cap (\prod_{j \in J} G_j) \leq Z_{\mathfrak{U}}(G).$$

2.4 Mutually permutable products

In this section results on mutually permutable products are presented. Most of them will be useful in Chapter 3 and Chapter 4.

Lemma 2.4.1. *Let a group $G = AB$ be the mutually permutable product of subgroups A and B. Then $G/N = (AN/N)(BN/N)$ is the mutually permutable product of subgroups AN/N and BN/N.*

Lemma 2.4.2. *[7, Lemma 4.1.37] Let the group $G = NB$ be a mutually permutable product of subgroups N and B. Suppose that N is normal in G and $N \cap B = 1$. Then B acts as a group of power automorphisms on N.*

Beidleman and Heineken proved the following structural results of mutually permutable products.

Lemma 2.4.3. *[17, Lemma 1((ii), (vi)and (vii))] Let a group $G = AB$ be the mutually permutable product of subgroups A and B. Then*
(i) if U is a normal subgroup of G, then $(U \cap A)(U \cap B)$ is also a normal subgroup of G,
(ii) if N is a non-abelian normal subgroup of G, then
$\{N \cap A, N \cap B\} \subseteq \{N, 1\}$ and $N = (N \cap A)(N \cap B)$,
(iii) if N is a minimal normal subgroup of G, then $\{N \cap A, N \cap B\} \subseteq \{N, 1\}$.
(iv) if N is a normal subgroup of G such that $A \cap B \leq N$, then G/N is a totally permutable product of subgroups AN/N and BN/N.

The following result generalises Lemma 2.4.3(i) and (ii):

Lemma 2.4.4. *[6, Lemma 1(i) and Lemma 3] Let a group $G = G_1G_2...G_n$ be the pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. Then*
(i) If U is a normal subgroup of G, then $(U \cap G_1)(U \cap G_2)...(U \cap G_n)$ is a normal subgroup of G,
(ii) if M is non-abelian minimal normal subgroup of G, then either $M \cap G_i = 1$ or $M \leq G_i$ for all $i \in \{1, 2, ..., n\}$. Moreover there exists $j \in \{1, 2, ..., n\}$ such that $M \leq G_j$.

Lemma 2.4.5. *[17, Corollary 3] Let a group $G = AB$ be the mutually permutable product of subgroups A and B. If V is the maximal perfect normal subgroup of A, then V is a normal subgroup of G.*

Hence the soluble residuals of the factors A and B are normal subgroups of G, in a mutually permutable product $G = AB$. In fact this is true for any finite number of factors as the following result by Ballester-Bolinches *et al.* [6] shows.

Theorem 2.4.6. *[6, Lemma 2] Let a group $G = G_1G_2...G_n$ be the pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. Then the soluble residuals of the factors G_i are normal subgroups of G and their product is the soluble residual of G.*

Lemma 2.4.7. *[17, Lemma 2] Let a group $G = AB$ be the mutually permutable product of subgroups A and B and let N be a minimal normal subgroup of G. If $N \cap A = N \cap B = 1$, then $|N| = p$ for some prime p and either $N \leq C_G(A)$ or $N \leq C_G(B)$.*

For pairwise totally permutable products it has been shown that one of the factors contains a normal subgroup of the product for a product with a finite number n of factors (Theorem 2.2.1). For mutually permutable products this has been shown to be true when the number of factors $n = 2$ by Beidleman and Heineken [18]:

Lemma 2.4.8. *[18, Theorem 1] Let a group $G = AB$ be the mutually permutable product of subgroups A and B. Then $Core_G(A)Core_G(B) \neq 1$.*

The following result was proved by Bochtler and the result was presented in [7].

Theorem 2.4.9. *[7, Theorem 4.4.5] Let a group $G = AB$ be the mutually permutable product of subgroups A and B. Then the nilpotent residual $B^{\mathfrak{N}}$ of B normalizes A.*

Theorem 2.4.10. *[6, Lemma 1(iii)] Let a group $G = G_1G_2...G_n$ be a pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. Then G_i' is a subnormal subgroup of G for all $i \in \{1, 2, ..., n\}$.*

Bochtler and Hauck in [20] showed that each of the subgroups A and B possess subnormal subgroups of G when $G = AB$ is a mutually permutable product:

Theorem 2.4.11. *[20, Theorem 2] Let a group $G = AB$ be the mutually permutable product of subgroups A and B. Then there exists subgroups L and M with the following properties:*
(i) $A' \leq L \leq A$, $B' \leq M \leq B$,
(ii) $A \cap B \leq L \cap M$,
(iii) L *and* M *are subnormal in* G,
(iv) $G' \trianglelefteq LB \cap MA = LM$.

Lemma 2.4.12. *[2, Theorem 1] Let a group $G = AB$ be the mutually permutable product of supersoluble subgroups A and B. If $Core_G(A \cap B) = 1$, then G is supersoluble.*

Bochtler generalized Lemma 2.4.12 to any saturated formation containing $\mathfrak{U}$.

Theorem 2.4.13. *[7, Theorem 4.5.8] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Consider a group $G = AB$ which is the mutually permutable product of subgroups A and B. If $Core_G(A \cap B) = 1$, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.*

2.5 Products of groups and SC-groups

In this section results on groups in which chief factors are simple are presented.

Definition 2.5.1. *A group G is an SC-group if all its chief factors are simple.*

The SC-groups were introduced by Robinson [33]. The class of SC-groups contains $\mathfrak{U}$. In fact a supersoluble group is a soluble SC-group.

Theorem 2.5.2. *[7, Theorem 1.6.3] The class of all SC-groups is a formation which is neither closed under taking subgroups nor saturated.*

Robinson characterised SC-groups:

Theorem 2.5.3. *[33, Proposition 2.4] A group G is an SC-group if and only if its soluble residual $G^{\mathfrak{S}}$ is such that*
(i) $G/G^{\mathfrak{S}}$ *is supersoluble.*

(ii) $G^{\mathfrak{S}}/Z(G^{\mathfrak{S}})$ *is a direct product of* G*-invariant simple groups.*

(iii) there is a G*-admissible series in* $Z(G^{\mathfrak{S}})$ *with cyclic factors.*

In general if $\mathfrak{F}$ is a formation containing $\mathfrak{U}$, then it is not necessarily true that G is in $\mathfrak{F}$ if and only if A and B are in $\mathfrak{F}$. However even though the class of SC-groups is not a formation of soluble groups or a saturated formation, this class behaves nicely with respect to totally permutable products as the following result shows.

Theorem 2.5.4. *[7, Theorem 4.5.12] Let the group* $G = G_1G_2...G_n$ *be the pairwise totally permutable product of subgroups* $G_1, G_2, ..., G_n$*. Then* G *is an* SC*-group if and only if* G_i *is an* SC*-group for all* $i \in \{1, 2, ..., n\}$*.*

Proof. By Theorem 2.2.4 it is sufficient to show that if G is an SC-group, then G_i is an SC-group for all $i \in \{1, 2, ..., n\}$. Suppose the result is not true and let G be a minimal counterexample. Then all factor groups of G satisfy the hypothesis. Since the class of SC-groups is a formation it follows that G has a unique minimal normal subgroup N and N is simple. If G_i is supersoluble for all $i \in \{1, 2, ..., n\}$, then G is supersoluble by Theorem 2.2.4, a contradiction. So there is $i \in \{1, 2, ..., n\}$ such that $G_i^{\mathfrak{U}} \neq 1$. Assume that $G_1^{\mathfrak{U}} \neq 1$. So $N \leq G_1^{\mathfrak{U}}$. Hence N centralises $K = G_2G_3...G_n$. This means N is a minimal normal subgroup of G_1. Since G_1/N is an SC-group and N is simple, G_1 is an SC-group.
If there exists G_i, $i \in \{2, ..., n\}$ such that G_i is not supersoluble then $N \leq G_i^{\mathfrak{U}}$ and arguing as above, G_i is an SC-group. Hence for all $i \in \{1, 2, ..., n\}$ either G_i is an SC-group or G_i is supersoluble and the result follows. □

Since the normal product of two supersoluble groups is not supersoluble Theorem 2.5.4 does not hold for mutually permutable products in general. However the following result holds:

Theorem 2.5.5. *[6, Theorem 5] Let the group* $G = G_1G_2...G_n$ *be the pairwise mutually permutable product of subgroups* $G_1, G_2, ..., G_n$*. If* G *is an* SC*-group, then* G_i *is an* SC*-group for all* $i \in \{1, 2, ..., n\}$*.*

The converse of Theorem 2.5.5 needs a restriction on the intersection to hold for mutually permutable products when $n = 2$.

Theorem 2.5.6. *[9, Theorem 2] Let the group $G = AB$ be a mutually permutable product of subgroups A and B. Suppose $Core_G(A \cap B) = 1$. If A and B are SC-groups, then G is an SC-group.*

Theorem 2.5.6 was proved by Ballester-Bolinches, Cossey and Pedraza-Aguilera in [9] and also by Beidlemen and Heineken in [18] using a different approach.

In this chapter the relationship totally (and respectively mutually) permutable products and their factors in the framework of formation theory was shown. The key difference between totally and mutually permutable products is that totally permutable products behave nicely with respect to forming products in formations containing $\mathfrak{U}$, whereas for mutually permutable products it is not true in general. In the next chapter, since weakly totally permutable products are between totally and mutually permutable products, the question that Peter Hauck asked:
Which of the results on totally permutable products can be extended to weakly totally permutable products?
is partially answered.

Chapter 3

Weakly Totally Permutable Products and Formations

This chapter is the author's original work. In this chapter results on weakly totally permutable products are presented. In particular, Theorems 2.2.4, 2.3.2, 2.3.1 (respectively in the case when $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$) and 2.5.4 are extended to weakly totally permutable products for $n = 2$. The results on weakly totally permutable products are proved using results on totally (and respectively mutually) permutable products that were presented in Chapter 2. Most of the work in this chapter was published in the Journal of Algebra [28].

3.1 Results on Structure

In this section some structural results that extend results on totally permutable products to weakly totally permutable products are presented.

Lemma 3.1.1. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B.*
(i) If H and K are subgroups of G such that $A \cap B \leq H \leq A$ and
$A \cap B \leq K \leq B$, then the subgroup HK is a weakly totally permutable product of subgroups H and K.

(ii) $A \cap B$ is a nilpotent subnormal subgroup of G.

(iii) If N is a minimal normal subgroup of G such that $N \leq A \cap B$, then N is a cyclic group of order p for some prime p.

(iv) If A is a minimal normal subgroup of G, then $G = AB$ is the totally permutable product of subgroups A and B.

Proof. (i) By hypothesis $A \cap B \leq H$ and $A \cap B \leq K$ so $A \cap B \leq H \cap K$. Also $H \cap K \leq A$ and $H \cap K \leq B$ which implies that $H \cap K \leq A \cap B$. Therefore $H \cap K = A \cap B$.

Let U be a subgroup of H such that $U \leq H \cap K = A \cap B$. Then U permutes with every subgroup of B and hence every subgroup of K.

Let V be a subgroup of H such that $H \cap K \leq V$. Note that V is a subgroup of A and $A \cap B = H \cap K \leq V$. It follows that V permutes with every subgroup of B and hence every subgroup of K. The same is true if K and H are interchanged. Hence the result follows.

(ii) Let H be a subgroup of $A \cap B$. By definition H is a permutable subgroup and hence a subnormal subgroup of both A and B.

By Theorem 2.0.4, H is a subnormal subgroup of $AB = G$. So H is a subnormal subgroup of $A \cap B$. It follows that $A \cap B$ is a nilpotent subnormal group by Theorem 1.2.7.

(iii) Since $N \leq A \cap B$, a nilpotent group it follows that N is an abelian p-subgroup for some prime p. Let P be a Sylow p-subgroup of G. Then $N \cap Z(P) \neq 1$. Let $g \in N \cap Z(P)$ be an element of order p. So the subgroup $\langle g \rangle$ is permutable in A and in B. Hence $\langle g \rangle$ is normalised by all Sylow q-subgroups of both A and B, that is, $\langle g \rangle$ is normal in A and B. Therefore $N = \langle g \rangle$ as required.

(iv) By Lemma 2.4.3(iii) either $A \cap B = 1$ or $A \cap B = A$. If $A \cap B = 1$, then the result follows. If $A \cap B = A$, then $G = (A \cap B)B$ is the totally permutable product of subgroups $A \cap B$ and B. □

Lemma 3.1.1(ii) generalises Lemma 2.1.3. Totally and mutually permutable products behave nicely with respect to factor groups. However weakly totally permutable products do not have this property as the following example shows:

Example 3.1.2. *Let G be the direct product of C_3(cyclic group of order 3) with B, where B is an extraspecial group of order 27 and exponent 3 whose presentation is*

$$B = \langle x, y | x^3 = y^3 = 1, [x, y] = z, zx = xz, yz = zy \rangle.$$

Suppose $C_3 = \langle c \rangle$. Let $A = \langle z, cx \rangle$. Then A and B are weakly totally permutable. But if $N = \langle c \rangle$, AN/N and BN/N are not weakly totally permutable since $\langle y \rangle N \leq AN/N \cap BN/N$ does not permute with $\langle x \rangle N = \langle cx \rangle N \leq AN/N \cap BN/N$.

The author [28] showed that if the normal subgroup is a product of normal subgroups of G contained in the factors, then weakly totally permutable products behave nicely with respect to factor groups, as the following result shows.

Lemma 3.1.3. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B.*
(i) If M and N are normal subgroups of G such that $M \leq A$ and $N \leq B$, then $G/MN = (AN/MN)(BM/MN)$ is the weakly totally permutable product of subgroups AN/MN and BM/MN.
(ii) Let N be a normal subgroup of G. Set $X = N \cap A$ and $Y = N \cap B$. Then $G/XY = (AY/XY)(BX/XY)$ is the weakly totally permutable product of subgroups AY/XY and BX/XY.

Proof. (i) Observe that $AN/MN \cap BM/MN = (AN \cap BM)/MN$. Since $M \leq A$ and $N \leq B$ it follows that $(AN \cap BM)/MN = (AN \cap B)M/MN = (A \cap B)MN/MN$ by Dedekind's Identity. Let HMN/MN be a subgroup of AN/MN and let KMN/MN be a subgroup of BM/MN. Then there is a subgroup W of B such that $WMN/MN = KMN/MN$.
If HMN/MN is a subgroup of $(A \cap B)MN/MN$, then there is a subgroup V of $A \cap B$ such that $VMN/MN = HMN/MN$. Since V permutes with W it follows that HMN/MN permutes with KMN/MN. Suppose HMN/MN contains $(A \cap B)MN/MN$. Then there is a subgroup U of A such that $UMN/MN = HMN/MN$. Consider $Y = U(A \cap B)$. Then $YMN/MN = U(A \cap B)MN/MN = HMN/MN$. Since Y permutes with W it also follows that HMN/MN permutes with KMN/MN.
The same is true if A and B are interchanged. Hence the result follows.

(ii) XY is a normal subgroup of G by Lemma 2.4.3(i). Then
$AY/XY \cap BX/XY = (AY \cap BX)/XY = (A \cap B)XY/XY$. The result now follows arguing as in the proof of (i). □

Using Lemma 2.4.3(ii) and Lemma 3.1.3(ii) it follows that if a group $G = AB$ is a weakly totally permutable product of subgroups A and B, and has a non-abelian minimal normal subgroup N, then G/N is a weakly totally permutable product of subgroups AN/N and BN/N.

Lemma 3.1.4. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let N be a minimal normal subgroup of G of order p for some prime p. Then G belongs to $\mathfrak{F}$ if and only if G/N belongs to $\mathfrak{F}$.*

Proof. If G belongs to $\mathfrak{F}$, then G/N also belongs to $\mathfrak{F}$ by the definition of a formation. Suppose $G/N \in \mathfrak{F}$. Since N is a cyclic group of order p, $G/C_G(N)$ is abelian of exponent $p-1$. Hence $G/C_G(N) \in U(p) \subseteq F(p)$, where U and F are canonical local definitions of $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Let H/K be a chief factor of G such that $N \leq K < H$. Then since G/N belongs to $\mathfrak{F}$ it follows that $G/C_G(H/K) \in F(q)$ for all primes q dividing $|H/K|$. Hence G belongs to $\mathfrak{F}$. □

Lemma 3.1.5. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then G belongs to $\mathfrak{F}$ if and only if A and B belong to $\mathfrak{F}$.*

Proof. By Theorem 2.4.13 if $\mathrm{Core}_G(A \cap B) = 1$, then G belongs to $\mathfrak{F}$ if and only if A and B belong to $\mathfrak{F}$. So assume that there is a minimal normal subgroup N of G such that
$N \leq \mathrm{Core}_G(A \cap B)$. By Lemma 3.1.1(iii) N is cyclic of order p for some prime p. Arguing by induction on $|G|$ and by Lemma 3.1.3(i) it follows that G/N belongs to $\mathfrak{F}$ if and only if A/N and B/N belong to $\mathfrak{F}$. Since N is a minimal normal subgroup of both subgroups A and B it follows from Lemma 3.1.4 that G belongs to $\mathfrak{F}$ if and only if A and B belong to $\mathfrak{F}$ as required. □

Lemma 3.1.5 generalises Theorems 2.2.4 and 2.2.6 to weakly totally permutable products when $n = 2$.

Lemma 3.1.6. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If E and F are $\mathfrak{F}$-projectors of A and B respectively, then $A \cap B$ is a subgroup of both E and F.*

Proof. Let E be an $\mathfrak{F}$-projector of A. Then $H = E(A \cap B)$ is the totally permutable product of subgroups E and $A \cap B$. Since $A \cap B$ is nilpotent (see Lemma 3.1.1), $H \in \mathfrak{F}$ by Lemma 3.1.5. Since E is $\mathfrak{F}$-maximal in A it follows that $H = E$ and hence $A \cap B$ is a subgroup of E. Analogously F contains $A \cap B$. □

Lemma 3.1.7. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $A^{\mathfrak{F}}, B^{\mathfrak{F}} \leq G^{\mathfrak{F}}$.*

Proof. Suppose the lemma is not true and let G be a minimal counterexample. Let $M = (G^{\mathfrak{F}} \cap A)(G^{\mathfrak{F}} \cap B) \neq 1$. By Lemma 3.1.3(ii) and Lemma 2.4.3(ii), G/M is the weakly totally permutable product of subgroups AM/M and BM/M. By the choice of G it follows that $A^{\mathfrak{F}}M \leq G^{\mathfrak{F}}M = G^{\mathfrak{F}}$, a contradiction.
Suppose $M = 1$. Let H/K be a chief factor of G such that $1 \leq K < H \leq G^{\mathfrak{F}}$. Since G/K is the weakly totally permutable product of subgroups AK/K and BK/K and $H/K \cap AK/K = H/K \cap BK/K = 1$ it follows that $|H/K| = p$ for some prime p by Lemma 2.4.7. Hence $G/C_G(H/K) \in U(p) \subseteq F(p)$, where U and F are canonical local definitions of $\mathfrak{U}$ and $\mathfrak{F}$, respectively. Let H/K be a chief factor of G such that $G^{\mathfrak{F}} \leq K < H$. Then since $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ it follows that $G/C_G(H/K) \in F(q)$ for all primes q dividing $|H/K|$. Hence G belongs to $\mathfrak{F}$ and by Lemma 3.1.5 both A and B belong to $\mathfrak{F}$ our final contradiction.
The same is true if A and B are interchanged. □

Lemma 3.1.8. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If B is supersoluble, then $G^{\mathfrak{U}} = A^{\mathfrak{U}}$.*

Proof. Suppose the lemma is not true and let G be a minimal counterexample. So assume that $G^{\mathfrak{U}}N = A^{\mathfrak{U}}N$ for any minimal normal subgroup N of G such that $N \leq A$ or $N \leq B$. By Theorem 2.4.13 if $\mathrm{Core}_G(A \cap B) = 1$, then $G^{\mathfrak{U}} = A^{\mathfrak{U}}$. Assume that there is a minimal normal subgroup N of G contained in $\mathrm{Core}_G(A \cap B)$. By Lemma 3.1.3(i) and the choice of G it follows that $G^{\mathfrak{U}}N = A^{\mathfrak{U}}N$. It also follows that $G^{\mathfrak{U}} \leq A$ and hence $A^{\mathfrak{U}}$ is a normal subgroup of $G^{\mathfrak{U}}$ by

Lemma 3.1.7 since $A^{\mathfrak{U}}$ is a normal subgroup of A. Let M be a minimal normal subgroup of G contained in $G^{\mathfrak{U}}$. By Lemma 3.1.3(i) $G^{\mathfrak{U}}M = G^{\mathfrak{U}} = A^{\mathfrak{U}}M$.

Let $A^{\mathfrak{S}} \neq 1$ be the soluble residual of A. By Theorem 2.4.6 $A^{\mathfrak{S}}$ is a normal subgroup of G. Since $A/A^{\mathfrak{U}} \in \mathfrak{U} \subseteq \mathfrak{S}$ it follows that $A^{\mathfrak{S}} \leq A^{\mathfrak{U}}$. Hence $G^{\mathfrak{U}} = A^{\mathfrak{S}}A^{\mathfrak{U}} = A^{\mathfrak{U}}$ which is a contradiction.

So A is soluble. Since B is supersoluble it follows that G is soluble by Theorem 2.4.6. Hence M is abelian. Now $G^{\mathfrak{U}}/A^{\mathfrak{U}} = MA^{\mathfrak{U}}/A^{\mathfrak{U}} \cong M/(M \cap A^{\mathfrak{U}})$ is abelian. It follows that $(G^{\mathfrak{U}})' \leq A^{\mathfrak{U}}$. If $(G^{\mathfrak{U}})' \neq 1$, then $G^{\mathfrak{U}} = A^{\mathfrak{U}}(G^{\mathfrak{U}})' = A^{\mathfrak{U}}$ because $(G^{\mathfrak{U}})'$ is a normal subgroup of G. So $G^{\mathfrak{U}}$ must be abelian.

Let A_1 be an $\mathfrak{U}$-projector of A by Theorem 1.8.5. Then $A \cap B \leq A_1$ by Lemma 3.1.6 and $G = A^{\mathfrak{U}}A_1B = G^{\mathfrak{U}}(A_1B)$. Since A_1B is the weakly totally permutable product of subgroups A_1 and B it follows that A_1B is supersoluble by Lemma 3.1.5. Let F be an $\mathfrak{U}$-maximal subgroup of G such that $A_1B \leq F$. Since $G = G^{\mathfrak{U}}F$, F is an $\mathfrak{U}$-projector of G by Lemma 1.8.7. By Theorem 1.8.9, $G^{\mathfrak{U}} \cap F = 1$. Hence $G^{\mathfrak{U}} = A^{\mathfrak{U}}(A_1B \cap G^{\mathfrak{U}}) = A^{\mathfrak{U}}$ which is our final contradiction. □

Lemma 3.1.9. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B^{\mathfrak{U}}] = [A^{\mathfrak{U}}, B] = 1$.*

Proof. Suppose the lemma is not true and let (G, B) be a counterexample with $|G| + |B|$ minimal.

Firstly it is argued that B is supersoluble. Let $D = A \cap B$. If for all $x \in B$, it implies that $D\langle x\rangle$ is a proper subgroup of G, then by the choice of G, $B = \langle D\langle x\rangle \mid x \in B\rangle$ centralises $A^{\mathfrak{U}}$, a contradiction. So assume that $B = D\langle x\rangle$, which is a totally permutable product of subgroups D and $\langle x\rangle$. Since D is nilpotent and $\langle x\rangle$ is cyclic, B is supersoluble by Lemma 3.1.5.

By Lemma 3.1.8, $G^{\mathfrak{U}} = A^{\mathfrak{U}}$. Let N be a minimal normal subgroup of G contained in $A^{\mathfrak{U}}$. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N by Lemma 3.1.3(i). Since BN/N is supersoluble, it follows that $BN/N \leq C_{G/N}(G^{\mathfrak{U}}N/N)$ by the choice of G and $[B, A^{\mathfrak{U}}] \leq N$.

If $A \cap B = B$, then $A = G$ and B are totally permutable and a contradiction follows from Lemma 2.3.3. Let M be a maximal subgroup of B such that $A \cap B \leq M$. Then AM is the weakly totally permutable product of subgroups A and M. So $(AM)^{\mathfrak{U}} = A^{\mathfrak{U}}$. Also $M \leq C_G(A^{\mathfrak{U}})$ by the choice of (G, B). If M_1 and M_2 are maximal subgroups of B such that $A \cap B \leq M_1$ and $A \cap B \leq M_2$, then $B = \langle M_1, M_2\rangle \leq C_G(A^{\mathfrak{U}})$ which is a contradiction. So B has a unique maximal subgroup M such that $A \cap B \leq M$ and $M \leq C_G(A^{\mathfrak{U}})$.

Let A_1 be a $\mathfrak{U}$-projector of A by Theorem 1.8.5. Then $A \cap B \leq A_1$. By Lemma 3.1.5, A_1B is supersoluble and $G = (A_1B)A^{\mathfrak{U}}$. Let U be a $\mathfrak{U}$-maximal subgroup of G containing A_1B. Then $G = UA^{\mathfrak{U}}$. So $Z_{\mathfrak{U}}(G) = C_U(A^{\mathfrak{U}})$ by Theorem 1.8.10, where $Z_{\mathfrak{U}}(G)$ is the $\mathfrak{U}$-hypercentre of G. This implies $M \leq Z_{\mathfrak{U}}(G)$. It is now to be shown that $N \leq Z_{\mathfrak{U}}(G)$.

By Theorem 2.4.9 B is a normal subgroup of $BA^{\mathfrak{U}}$ and so $[B, A^{\mathfrak{U}}] \leq B$. Suppose $N \cap B = 1$. Then $[B, A^{\mathfrak{U}}] \leq B \cap N = 1$, a contradiction. Hence $N \leq A \cap B \leq M \leq Z_{\mathfrak{U}}(G)$ and $Z_{\mathfrak{U}}(G/N) = Z_{\mathfrak{U}}(G)/N$.

But $G/N = (UN/N)(A^{\mathfrak{U}}/N)$ and $(G/N)^{\mathfrak{U}} = A^{\mathfrak{U}}/N$. Let U_1/N be a $\mathfrak{U}$-maximal subgroup of G/N containing UN/N. It follows that $Z_{\mathfrak{U}}(G)/N = Z_{\mathfrak{U}}(G/N) = C_{U_1/N}(A^{\mathfrak{U}}/N)$ by Theorem 1.8.10. By the choice of G it also follows that $B/N \leq Z_{\mathfrak{U}}(G)/N$ and so $B \leq Z_{\mathfrak{U}}(G)$ which is our final contradiction.

The same is true if A and B are interchanged. □

Lemma 3.1.9 generalises Corollary 2.1.12.

Lemma 3.1.10. *Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $A^{\mathfrak{F}}$ and $B^{\mathfrak{F}}$ are normal subgroups of G.*

Proof. Since $A/A^{\mathfrak{U}} \in \mathfrak{U} \subseteq \mathfrak{F}$, $A^{\mathfrak{F}} \leq A^{\mathfrak{U}}$. By Lemma 3.1.9, B centralises $A^{\mathfrak{U}}$ and hence $B \leq C_G(A^{\mathfrak{F}})$. Since $A^{\mathfrak{F}}$ is normal in A it means that $A^{\mathfrak{F}}$ is a normal subgroup of G.

Analogously $B^{\mathfrak{F}}$ is normal in G.

□

Open Question 3.1.11. *Can Theorem 2.3.3 be extended to weakly totally permutable products?*

3.2 Weakly Totally Permutable Products and Formation Subgroups

In this section Theorems 2.2.4, 2.3.2 and 2.3.1 (respectively in the case when $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$) are extended to weakly totally permutable products. These results are relationships between $\mathfrak{F}$-residuals (respectively $\mathfrak{F}$-projectors) of subgroups and that of the product, where $\mathfrak{F}$ is a saturated formation.

Theorem 3.2.1. *[28, Theorem 1] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$.*

Proof. The proof of the theorem is by induction on $|G|$. Using Lemma 3.1.5 assume that $G^{\mathfrak{F}} \neq 1$ and $A^{\mathfrak{F}}B^{\mathfrak{F}} \neq 1$. Let N be a minimal normal subgroup of G contained in $A^{\mathfrak{F}}$ using Lemma 3.1.10. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N and so $G^{\mathfrak{F}}N = A^{\mathfrak{F}}B^{\mathfrak{F}}N = A^{\mathfrak{F}}B^{\mathfrak{F}}$. Hence $G^{\mathfrak{F}} \leq A^{\mathfrak{F}}B^{\mathfrak{F}}$. By Lemma 3.1.7 $A^{\mathfrak{F}}B^{\mathfrak{F}} \leq G^{\mathfrak{F}}$. Hence the result now follows. □

Theorem 3.2.1 extends part of Theorem 2.3.1 (when $\mathfrak{F}$ is a saturated formation) to weakly totally permutable products when $n = 2$.

Theorem 3.2.2. *[28, Theorem 2] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A_1 and B_1 are $\mathfrak{F}$-projectors of A and B respectively, then A_1B_1 is an $\mathfrak{F}$-projector of G.*

Proof. Suppose the theorem is not true and let $G = AB$ be a minimal counterexample. So $A \neq 1$ and $B \neq 1$. By Lemmas 3.1.5 and 3.1.10 it can be assumed that there exists a minimal normal subgroup N of G such that $N \leq A$. Then $G/N = (A/N)(BN/N)$ is the weakly totally permutable product of subgroups A/N and BN/N. This means that $(A_1N/N)(B_1N/N) = (A_1B_1)N/N$ is an $\mathfrak{F}$-projector of G/N by the choice of G. Consider $C = (A_1B_1)N$. Suppose C is a proper subgroup of G. Since $A \cap B$ is contained in both A_1 and B_1, the subgroup $(A_1N)B_1$ is the weakly totally permutable product of subgroups A_1N and B_1. Also A_1 is an $\mathfrak{F}$-projector of A_1N by Lemma 1.8.7. This implies that A_1B_1 is an $\mathfrak{F}$-projector of C by the minimality of G. Therefore A_1B_1 is an $\mathfrak{F}$-projector of G using Theorem 1.8.6 which is a contradiction.
Hence $G = (A_1B_1)N$. By Theorem 3.2.1 $A_1B_1 \in \mathfrak{F}$. Since $G/N \in \mathfrak{F}$, $G^{\mathfrak{F}} \leq N$. Also $G^{\mathfrak{F}}$ is a normal subgroup of G so $G^{\mathfrak{F}} = 1$ or $G^{\mathfrak{F}} = N$. If $G^{\mathfrak{F}} = 1$, then $A_1 = A$ and $B_1 = B$ by Theorem 3.2.1, a contradiction. Hence $G^{\mathfrak{F}} = N$. Let F be an $\mathfrak{F}$-maximal subgroup of G containing A_1B_1. It follows that F is an $\mathfrak{F}$-projector of G by Lemma 1.8.7. Also $F = F \cap A_1B_1N = (A_1(F \cap N))B_1$ which is the weakly totally permutable product of subgroups $A_1(F \cap N)$ and B_1. Since $F \in \mathfrak{F}$

it follows that $A_1(F \cap N) \in \mathfrak{F}$ by Theorem 3.2.1. But A_1 is $\mathfrak{F}$-maximal in A, so $A_1 = A_1(F \cap N)$ and $F = (A_1(F \cap N))B_1 = A_1B_1$ is an $\mathfrak{F}$-projector of G, our final contradiction. □

Theorem 3.2.2 generalises Theorem 2.3.2 for $n = 2$.

Theorem 3.2.3. *[28, Theorem 3] Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A and B belong to $\mathfrak{F}$, then G also belongs to $\mathfrak{F}$.*

Proof. Suppose the theorem is not true and let G be a counterexample with $|G| + |A| + |B|$ minimal. So $A, B \in \mathfrak{F}$ but $G \notin \mathfrak{F}$. By Theorem 3.2.1 either A or B is not supersoluble. Assume $A^{\mathfrak{U}} \neq 1$. By Lemma 3.1.9, B centralises $A^{\mathfrak{U}}$. It follows that $A^{\mathfrak{U}}$ centralises $\langle B^G \rangle$ since $A^{\mathfrak{U}}$ is normal in G. Let A_1 be a $\mathfrak{U}$-projector of A by Theorem 1.8.5. Then $A \cap B \leq A_1$ by Lemma 3.1.6 and $A = A_1A^{\mathfrak{U}}$. Consider A_1B, the weakly totally permutable product of subgroups A_1 and B. It means that $A_1 \in \mathfrak{U} \subseteq \mathfrak{F}$ and $B \in \mathfrak{F}$ and $A_1B \in \mathfrak{F}$ since
$|A_1B| + |A_1| + |B| < |G| + |A| + |B|$. Since $\langle B^G \rangle$ is a normal subgroup of G, A acts on $\langle B^G \rangle$ by conjugation. Let $Z = [\langle B^G \rangle]A$ be the semidirect product of $\langle B^G \rangle$ and A with respect to this action. So G is a quotient group of Z by Lemma 2.0.6. It is sufficient to show that Z belongs to $\mathfrak{F}$.
Since $A^{\mathfrak{U}}$ centralise $\langle B^G \rangle$, $A^{\mathfrak{U}}$ is normal in Z. So $Z/A^{\mathfrak{U}} = [\langle B^G \rangle]A_1A^{\mathfrak{U}}/A^{\mathfrak{U}}$ is isomorphic to $[\langle B^G \rangle](A_1/(A_1 \cap A^{\mathfrak{U}}))$, a quotient group of $[\langle B^G \rangle]A_1$.
Since $A^{\mathfrak{U}} \leq C_G(B)$ it follows that $\langle B^G \rangle = \langle B^A \rangle = \langle B^{A_1} \rangle$. There exists an epimorphism $\varphi : [\langle B^G \rangle]A_1 \to BA_1$ by Lemma 2.0.6. So $[\langle B^G \rangle]A_1/\langle B^{A_1} \rangle$ which is isomorphic to $A_1 \in \mathfrak{U} \subseteq \mathfrak{F}$ and $[\langle B^G \rangle]A_1/\operatorname{Ker} \varphi \cong BA_1 \in \mathfrak{F}$. Hence $[\langle B^G \rangle]A_1/(\operatorname{Ker} \varphi \cap \langle B^{A_1} \rangle) = [\langle B^G \rangle]A_1 \in \mathfrak{F}$. Therefore $Z/A^{\mathfrak{U}} \in \mathfrak{F}$ and so $Z/\langle B^G \rangle \cong A \in \mathfrak{F}$. By Lemma 2.0.6, $\langle B^G \rangle \cap A^{\mathfrak{U}} = 1$ which implies $Z/(\langle B^G \rangle \cap A^{\mathfrak{U}}) = Z \in \mathfrak{F}$. The result follows since G is a quotient group of Z. □

Open Question 3.2.4. *Do Theorems 3.2.1, 3.2.2 and 3.2.3 hold for pairwise weakly totally permutable products?*

3.3 Weakly Totally Permutable Products and SC-groups

In this section Theorem 2.5.4 is extended to weakly totally permutable products for $n = 2$.

Theorem 3.3.1. *Let the group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then G is an SC-group if and only if A and B are SC-groups.*

Proof. The argument is by induction on $|G|$. By Theorem 2.5.5 it is sufficient to prove that if A and B are SC-groups, then G is an SC-group. By Theorem 2.5.6 it can be assumed that there exists a minimal normal subgroup N such that $N \leq \mathrm{Core}_G(A \cap B)$. Since G/N satisfies the hypothesis, it follows that G/N is an SC-group by induction. By Lemma 3.1.1(iii) N is a cyclic group of order p for some prime p. Hence G is an SC-group as required. □

Open Question 3.3.2. *Does Thereom 3.3.1 hold for pairwise weakly totally permutable products?*

It has been shown that some results on totally permutable products also hold for weakly totally permutable products in the framework of formations when the number of factors is two. Dual to the concept of formations is that of Fitting classes. In Chapter 4 products of finite groups are studied in the framework of Fitting classes.
The question by Peter Hauck:
Which of the results on totally permutable products can be extended to weakly totally permutable products?
is also partially answered.

Chapter 4

Products of Finite Groups and Fitting Classes

In this chapter results on Fitting classes and how they relate to products of finite groups are presented. In Chapter 3 it was shown that saturated formations behave nicely with respect to weakly totally permutable products. An attempt is made to extend results that hold for totally permutable products to weakly totally permutable products. Fischer classes containing $\mathfrak{U}$ were proved to behave nicely with respect to forming products in totally permutable products. It is shown that a particular Fischer class, $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$ and $\mathfrak{N}$ is the class of all nilpotent groups, also behave nicely with respect forming products in weakly totally permutable products.

4.1 Fitting Classes

In this section results on the structure of Fitting classes are presented.

Definition 4.1.1. *A non-empty class $\mathfrak{F}$ is a Fitting class if and only if the following two conditions are satisfied:*
(i) If $G \in \mathfrak{F}$ and $N \triangleleft G$, then $N \in \mathfrak{F}$;
(ii) If $M, N \triangleleft G = MN$ with M and N in $\mathfrak{F}$, then $G \in \mathfrak{F}$.

A result which defines an $\mathfrak{F}$-radical is presented below.

Lemma 4.1.2. *[23, II, Lemma 2.9] Let $\mathfrak{X}$ be an N_0-closed class and G be a finite group. Then the set $\mathfrak{S} = \{N \ sn \ G : N \in \mathfrak{X}\}$, partially ordered by inclusion, has a unique maximal element, denoted by $G_{\mathfrak{X}}$ and called the $\mathfrak{X}$-radical of G. It is a characteristic subgroup of G and if $\mathfrak{X}$ is a Fitting class and $K \ sn \ G$, then $K_{\mathfrak{X}} = K \cap G_{\mathfrak{X}}$.*

So the $\mathfrak{F}$-radical, $G_{\mathfrak{F}}$ of G is the unique maximal normal $\mathfrak{F}$-subgroup of G and this is the join of all subnormal $\mathfrak{F}$-subgroups of G . Hence $G_{\mathfrak{N}} = F(G)$.

Definition 4.1.3. *Let $\mathfrak{F}$ be a Fitting class and $\mathfrak{C}$ a class of finite groups. Define*

$$\mathfrak{F} \diamond \mathfrak{C} = (G : G/G_{\mathfrak{F}} \in \mathfrak{C})$$

and call $\mathfrak{F} \diamond \mathfrak{C}$ the Fitting product of $\mathfrak{F}$ with $\mathfrak{C}$.

The following result shows that the Fitting product defined above is in fact a Fitting class when $\mathfrak{C}$ is a Fitting class.

Theorem 4.1.4. *[23, IX, Theorem 1.12(a)] Let $\mathfrak{F}$ and $\mathfrak{C}$ be a Fitting classes. Then $\mathfrak{F} \diamond \mathfrak{C}$ is a Fitting class.*

There exists a special type of Fitting products which is defined in the next line. A group $G \in \mathfrak{X} \diamond \mathfrak{X} = \mathfrak{X}\mathfrak{X} = \mathfrak{X}^2$ where $\mathfrak{X}$ is a Fitting class of groups, is called a meta-$\mathfrak{X}$ group. An example of this type of group which forms a Fitting class is a metanilpotent group. This is a group G such that $G/F(G)$ is nilpotent and denoted by $\mathfrak{N} \diamond \mathfrak{N} = \mathfrak{N}^2$. Since the derived group of a supersoluble group is nilpotent, it follows that $\mathfrak{U} \subseteq \mathfrak{N}^2$.
In general Fitting classes are not R_0-closed. The following result can be substituted for R_0-closure. Its known as the quasi-R_0 Lemma.

Theorem 4.1.5. *(The quasi-R_0 Lemma)[23, IX, Lemma 1.13]*
Let N_1 and N_2 be normal subgroups of a group G such that $N_1 \cap N_2 = 1$ and G/N_1N_2 is nilpotent, and let $\mathfrak{F}$ be a Fitting class containing G/N_1. Then $G \in \mathfrak{F}$ if and only if $G/N_2 \in \mathfrak{F}$.

A special type of a Fitting class is defined below.

Definition 4.1.6. *A Fischer class is a Fitting class such that if $K \lhd G \in \mathfrak{F}$ and H/K is nilpotent subgroup of G/K, then $H \in \mathfrak{F}$.*

An example of a Fischer class is given in [23, IX, Examples (3.7)(c)(2)](pg. 604) which is the Fitting product $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class. From this it follows that $\mathfrak{N}^2$ is a Fischer class containing $\mathfrak{U}$.

4.2 Totally Permutable Products and Fitting Classes

It is not known if, in general, Fitting classes containing the class of all supersoluble groups are closed under the formation of totally permutable products. In this section some results on totally permutable products and Fitting classes are presented. This is mainly the work of Hauck, Martínez-Pastor and Pérez-Ramos [24]. They proved that Fischer classes behave nicely with respect to forming totally permutable products. The arguments presented are based on their original proofs in [24] and [25]. In the following result Propositions 1 and 2 of [24] are combined.

Lemma 4.2.1. *[24, Propositions 1 and 2] Let $\mathfrak{h}$ be a subgroup-closed saturated formation and let*

$$\mathfrak{T}_{\mathfrak{h}} = (G = AB; A \text{ and } B \text{ are totally permutable, } A \cap B \leq Z_{\mathfrak{h}}(G)).$$

Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$. Suppose that the following cases hold:

Case 1 : *there exists $G = AB \in \mathfrak{T}_{\mathfrak{h}}$ with $A, B \in \mathfrak{F}$ but $G \notin \mathfrak{F}$, and that among all these groups in $\mathfrak{T}_{\mathfrak{h}}$, let (G, A, B) be a group with least $|G| + |A||B|$, or*

Case 2 : *there exists $G = AB \in \mathfrak{T}_{\mathfrak{h}}$ with $G \in \mathfrak{F}$, but not both of A and B in $\mathfrak{F}$, and among all these groups in $\mathfrak{T}_{\mathfrak{h}}$, let (G, A, B) a group with least $|G| + |A||B|$.*

Then, after interchanging the roles of A and B if neccessary, the following holds:

1. $A/A^{\mathfrak{N}}$ *is a cyclic* p*-group for a prime* p *and* $A^{\mathfrak{N}}$ *is a non-trivial normal subgroup of* G*.*

2. B *is a normal abelian* p'*-subgroup and* p *divides* $q-1$ *for all prime divisors* q *of* $|B|$*.*

3. $B = [A, B]$*.*

4. $A \cap B = A^{\mathfrak{N}} \cap B \leq Z_{\mathfrak{U}}(G)$ *but* $A \cap B \nleq Z_{\infty}(G)$*.*

5. A *acts as a group of power automorphisms on* B*.*

If $\mathfrak{U} \subseteq \mathfrak{h}$*, then condition 2 can be replaced by*

2′. B *is a normal cyclic* q*-subgroup, where* q *is a prime such that* p *divides* $q-1$*.*

Proof. Since $\mathfrak{h}$ is subgroup closed, it follows that $Z_{\mathfrak{h}}(G) \cap M \leq Z_{\mathfrak{h}}(M)$ for every subgroup M of G. So if $A_1 \leq A$ and $B_1 \leq B$, then $A_1B_1 \in \mathfrak{T}_{\mathfrak{h}}$.
The proof is split into the following steps.

(1) *Without loss of generality it may be assumed that* B *is a nilpotent group and that* A *is not nilpotent. Moreover,* $[B, A^{\mathfrak{N}}] = 1$*.*

By Theorem 2.3.3, $[B, A^{\mathfrak{N}}] = [A, B^{\mathfrak{N}}] = 1$. Suppose that A and B are both nilpotent.
If Case 1 holds, then G would be supersoluble by Theorem 3.2.3 which contradicts the choice of (G, A, B).
By the choice of A and B Case 2 does not hold.
Hence A and B cannot both be nilpotent.
Suppose that $A^{\mathfrak{N}} \neq 1$ and $B^{\mathfrak{N}} \neq 1$. Note that $A^{\mathfrak{N}}$ cannot be central in A and $B^{\mathfrak{N}}$ cannot be central in B otherwise A and B will be nilpotent. Then since $[A^{\mathfrak{N}}, B] = [A, B^{\mathfrak{N}}] = 1$, it follows that $B \leq C_G(A^{\mathfrak{N}}) < G$ and $A \leq C_G(B^{\mathfrak{N}}) < G$. Hence $C_G(A^{\mathfrak{N}}) = B(A \cap C_G(A^{\mathfrak{N}}))$ is a totally permutable product of the subgroups B and $A \cap C_G(A^{\mathfrak{N}})$.

Assume that Case 1 holds. Then since $A \cap C_G(A^{\mathfrak{N}})$ is a normal subgroup of $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, it follows that $A \cap C_G(A^{\mathfrak{N}}) \in \mathfrak{F}$. So $C_G(A^{\mathfrak{N}}) \in \mathfrak{F}$ since
$|G| + |A| + |B| < |C_G(A^{\mathfrak{N}})| + |A \cap C_G(A^{\mathfrak{N}})| + |B|$. Analogously $C_G(B^{\mathfrak{N}}) \in \mathfrak{F}$ and so $G = C_G(A^{\mathfrak{N}})C_G(B^{\mathfrak{N}}) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ because it is a product of two normal $\mathfrak{F}$-subgroups.
Assume now that Case 2 holds. Then $C_G(A^{\mathfrak{N}})$ is an $\mathfrak{F}$-subgroup because it is a normal subgroup of G. By the choice of (G, A, B), $B \in \mathfrak{F}$. Analogously $A \in \mathfrak{F}$, contradicting the choice of (G, A, B).
Therefore A and B cannot be both non-nilpotent. Hence (1) follows.

(2) $G = \langle A^G \rangle = A[A, B]$, *$\langle B^G \rangle \cap A$ is nilpotent and* $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$.

Note first that $[B, A^{\mathfrak{N}}] = 1$ implies that $[\langle B^G \rangle, A^{\mathfrak{N}}] = 1$ since $A^{\mathfrak{N}}$ is a normal subgroup of G. Now

$$(\langle B^G \rangle \cap A)/(\langle B^G \rangle \cap A^{\mathfrak{N}}) \cong (\langle B^G \rangle \cap A)A^{\mathfrak{N}}/A^{\mathfrak{N}} \in \mathfrak{N}$$

and

$$\langle B^G \rangle \cap A^{\mathfrak{N}} \leq Z(\langle B^G \rangle \cap A),$$

so $\langle B^G \rangle \cap A$ is nilpotent. Consequently $\langle B^G \rangle = B(\langle B^G \rangle \cap A)$ is a totally permutable product of two nilpotent groups $\langle B^G \rangle \cap A$ and B. Then $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$ by Theorem 3.2.3.
Assume now that $\langle A^G \rangle$ is a proper subgroup of G. Then $\langle A^G \rangle = A(\langle A^G \rangle \cap B)$ is a totally permutable product of subgroups A and $\langle A^G \rangle \cap B$.
Assume Case 1 holds. Then $\langle A^G \rangle \cap B$ is a normal subgroup of B and so $\langle A^G \rangle \cap B \in \mathfrak{F}$. By the choice of (G, A, B) it follows that $\langle A^G \rangle \in \mathfrak{F}$. Since $\langle B^G \rangle \in \mathfrak{U}$, it follows that $G = \langle A^G \rangle \langle B^G \rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction.
Assume Case 2 holds. Then $\langle A^G \rangle \in \mathfrak{F}$ because $\langle A^G \rangle$ is a normal subgroup of G and by the choice of (G, A, B), $A \in \mathfrak{F}$, a contradiction. Hence $G = \langle A^G \rangle$.

(3) *There exists a prime number p such that $G = A^{\mathfrak{N}} A_p \langle B^G \rangle$, where A_p is a Sylow p-subgroup of A.*

Assume that $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is a proper subgroup of G for all primes q, where A_q denotes a Sylow q-subgroup of A. Then

$$A^{\mathfrak{N}}A_q\langle B^G\rangle = A^{\mathfrak{N}}A_q\langle B^G\rangle \cap AB = A^{\mathfrak{N}}A_q(\langle B^G\rangle \cap A)B.$$

Now $[A^{\mathfrak{N}}A_q, B] \leq [A, B] \leq \langle B^G\rangle$ so $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is normalised by B and $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is normalised by A since $A/A^{\mathfrak{N}}$ is nilpotent and $A^{\mathfrak{N}}A_q/A^{\mathfrak{N}}$ is the normal Sylow q-subgroup of $A/A^{\mathfrak{N}}$ and so $A^{\mathfrak{N}}A_q$ is a normal in A. Hence $A^{\mathfrak{N}}A_q\langle B^G\rangle$ is a normal subgroup of G. The subgroups $X_q = A^{\mathfrak{N}}A_q(\langle B^G\rangle \cap A)$ and B are totally permutable. Assume that Case 1 holds. Then $X_q \in \mathfrak{F}$ because X_q is a normal subgroup of A and, by the choice of (G, A, B), it follows that $X_qB \in \mathfrak{F}$. Hence $G = \prod_{q\in\mathbb{P}} X_qB \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction of the assumption. Assume now that Case 2 holds. Then X_qB is a normal subgroup of G and so $X_qB \in \mathfrak{F}$. By the minimality of (G, A, B), $X_q \in \mathfrak{F}$ for all primes q. It follows that $A = \prod_{q\in\mathbb{P}} X_q \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction of the assumption.

(4) *Assume that Case 1 holds. Then for all primes $q \neq p$, $A^{\mathfrak{N}}A_q[A, B]$ is a normal $\mathfrak{F}$-subgroup of G, where A_q is a Sylow q-subgroup of A, but $A^{\mathfrak{N}}A_p[A_p, B] \notin \mathfrak{F}$.*

By (3) $G = \langle A^G\rangle = \langle A^B\rangle = A[A, B]$. Since $[A, B]$ is a normal subgroup of G, and $A^{\mathfrak{N}}A_q$ is a normal subgroup of A, it follows that $A^{\mathfrak{N}}A_q[A, B]$ is a normal subgroup of G. Now $A^{\mathfrak{N}}, \langle B^G\rangle \in \mathfrak{F}$ by (2) and so $A^{\mathfrak{N}}\langle B^G\rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$. Since $G/A^{\mathfrak{N}}\langle B^G\rangle$ is a p-group, it follows that $A^{\mathfrak{N}}A_q[A, B]$ is contained in $A^{\mathfrak{N}}\langle B^G\rangle$. Hence $A^{\mathfrak{N}}A_q[A, B]$ is an $\mathfrak{F}$-group.
Consider the normal subgroup $C = \langle (A^{\mathfrak{N}}A_p)^G\rangle$ of G. Since $[B, A^{\mathfrak{N}}] = 1$, it follows that $C = \langle (A^{\mathfrak{N}}A_p)^B\rangle = A^{\mathfrak{N}}A_p[A^{\mathfrak{N}}A_p, B] = A^{\mathfrak{N}}A_p[A_p, B]$. Suppose that $C \in \mathfrak{F}$. Then by (2), it follows that

$$G = \langle A^G\rangle = A[A, B] = C(\textstyle\prod_{q\neq p} A^{\mathfrak{N}}A_q[A, B]) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F},$$

contradicting the choice of (G, A, B).

(5) *Assume that Case 2 holds. Then for all primes* $q \neq p$, $A^{\mathfrak{N}}A_q \in \mathfrak{F}$, *where* A_q *is a Sylow* q*-subgroup of* A. *Moreover,* $A^{\mathfrak{N}}A_p \notin \mathfrak{F}$.

By (3) $X_q = A^{\mathfrak{N}}A_q \leq A^{\mathfrak{N}}\langle B^G\rangle = X_q\langle B^G\rangle$. It follows that $A^{\mathfrak{N}}\langle B^G\rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ because $A^{\mathfrak{N}} \in \mathfrak{F}$ and $\langle B^G\rangle$ is supersoluble by (2). Since $X_q\langle B^G\rangle$ is a proper subgroup of G, which is a totally permutable product of subgroups $A \cap X_q\langle B^G\rangle$ and B, it follows that $A \cap X_q\langle B^G\rangle \in \mathfrak{F}$ by the choice of (G, A, B). Then $X_q \in \mathfrak{F}$ because X_q is normal in $A \cap X_q\langle B^G\rangle$. Finally, if $X_p \in \mathfrak{F}$, then $A = \prod_{q \in \mathbb{P}} X_q \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, contradicting the assumption.

(6) $G = A^{\mathfrak{N}}A_p[A_p, B] = A^{\mathfrak{N}}A_pB$.

Let $C = A^{\mathfrak{N}}A_p[A_p, B] = \langle (A^{\mathfrak{N}}A_p)^G\rangle$. Assume that C is a proper subgroup of G. Then $C = C \cap (A^{\mathfrak{N}}A_pB) = A^{\mathfrak{N}}A_p(C \cap B)$ which is a totally permutable product of subgroups $A^{\mathfrak{N}}A_p$ and $C \cap B$.
Assume Case 1 holds. Then by the choice of (G, A, B), $C \in \mathfrak{F}$ which contradicts (4).
Assume now that Case 2 holds. Then $C \in \mathfrak{F}$ since C is a normal subgroup of G. By the choice of (G, A, B) it follows that $A^{\mathfrak{N}}A_p \in \mathfrak{F}$ which contradicts (5). Hence $C = G$. Since $C = A^{\mathfrak{N}}A_p[A_p, B] \leq A^{\mathfrak{N}}A_pB$, it also follows that $G = A^{\mathfrak{N}}A_pB$.

(7) $A/A^{\mathfrak{N}}$ *is a* p*-group. In particular,* $A = A^{\mathfrak{N}}A_p$.

By (6), $G = A^{\mathfrak{N}}A_pB$. If $A^{\mathfrak{N}}A_p < A$, then by the choice of (G, A, B) it follows that $G = A^{\mathfrak{N}}A_pB \in \mathfrak{F}$ if Case 1 holds, this is a contradiction and $A^{\mathfrak{N}}A_p \in \mathfrak{F}$ which contradicts (5) if Case 2 holds. So $A^{\mathfrak{N}}A_p = A$. Hence $A/A^{\mathfrak{N}}$ is a p-group.

(8) p *divides* $q-1$ *for all prime divisors* q *of* $|B|$ *different from* p. *Moreover,* $G = A^{\mathfrak{N}}A_pO_p(B)O_{p'}(B)$ *and* $O_{p'}(B)$ *is a normal subgroup of* G.

Firstly it is shown that there exists a prime q dividing $|B|$ with $p < q$. Suppose the contrary, that is, $p \geq q$ for all primes q dividing $|B|$. By Theorem 2.2.4 A_pB is supersoluble. It follows that $A_pO_p(B)$ is a normal subgroup of A_pB by Theorem 1.2.5 which implies that A_p is a subnormal subgroup of A_pB. Since $A^{\mathfrak{N}}$ is normal subgroup of G, it follows that $A^{\mathfrak{N}}A_p = A$ is a subnormal subgroup of $A^{\mathfrak{N}}A_pB = G$.

Assume Case 1 holds. Then since B is subnormal in G by (2), it follows that $G = AB \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, against the choice of (G, A, B).

If Case 2 holds, then $A \in \mathfrak{F}$ since it is a subnormal subgroup of $G \in \mathfrak{F}$ which again contradicts the choice of (G, A, B).

Let $\pi(B)$ be the set of all primes dividing $|B|$ and consider $\pi(B) \cup \{p\} = \{p_1, p_2, ..., p_t = p, p_{t+1}, ..., p_n\}$ with $p_1 < p_2 < ... < p_t = p < p_{t+1} < ... < p_n$. Denote by $\pi = \{p, p_{t+1}, ..., p_n\}$ and $\pi' = (\pi(B) \cup \{p\})\backslash\pi$. Since A_pB is a supersoluble group, $A_pO_\pi(B)$ is normalised by B by Theorem 1.2.5. Also A_p is normalised by $B_{\pi'}$. Then $[A_p, B] = [A_p, B_{\pi'}O_\pi(B)] = [A_p, B_{\pi'}][A_p, O_\pi(B)] \leq A_p[A_p, O_\pi(B)]$. By (6) $G = A^{\mathfrak{N}}A_p[A_p, B] = A^{\mathfrak{N}}A_p[A_p, O_\pi(B)]$ is contained in $A^{\mathfrak{N}}A_pO_\pi(B)$ and so $G = A^{\mathfrak{N}}A_pO_\pi(B)$. Assume that $AO_\pi(B) < G$. By the choice of (G, A, B) it follows that in Case 1 $G \in \mathfrak{F}$ and in Case 2 $A \in \mathfrak{F}$ which are both contradictions. Hence $B = O_\pi(B)$ and so $p \leq q$ for all primes $q \in \pi(B)$.

Now since $p < q$ for all primes $q \in \pi(B)\backslash\{p\}$ and A_pB is supersoluble, p divides $q-1$. It follows that $O_{p'}(B)$ is centralised by $A^{\mathfrak{N}}$ and normalised by A_p. Hence $O_{p'}(B)$ is a normal subgroup of $G = A^{\mathfrak{N}}A_pO_p(B)O_{p'}(B)$ since $O_{p'}(B)$ is normal in B.

(9) *B is a normal p'-subgroup of G. Moreover, $B = [A, B]$.*

Suppose that B is not a p'-subgroup. Then it means that $O_{p'}(B) < B$, that is, $AO_{p'}(B) < G$. By (8) $A^{\mathfrak{N}}O_{p'}(B)$ is a normal subgroup of G and $G/A^{\mathfrak{N}}O_{p'}(B)$ is a p-group. So $AO_{p'}(B)$ is a subnormal subgroup of G.

Assume that Case 1 holds. Then by the choice of (G, A, B) it follows that $AO_{p'}(B) \in \mathfrak{F}$. This means that

$$G = (AO_{p'}(B))\langle B^G\rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F},$$

since $AO_{p'}(B)$ is subnormal and $\langle B^G\rangle$ is normal in G. This contradicts the choice of (G, A, B).

Assume that Case 2 holds. Then since $AO_{p'}(B)$ is a subnormal subgroup of $G \in \mathfrak{F}$, it follows that $AO_{p'}(B) \in \mathfrak{F}$. By the minimality of (G, A, B) it also follows that $A \in \mathfrak{F}$, a contradiction.

Therefore $G = AO_{p'}(B)$ and $B = O_{p'}(B)$ is a normal p'-subgroup of G by the minimality of

(G, A, B).

Hence $[A, B] = [A_p, B] \leq B$ and $[A, B]$ is a normal subgroup of G and $[A, B] \in \mathfrak{N} \subseteq \mathfrak{F}$. By (2) $G = \langle A^G \rangle = A[A, B]$. Suppose that $[A, B]$ is a proper subgroup of B.

Assume that Case 1 holds. Then by the choice of (G, A, B) it follows that $G = A[A, B] \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, a contradiction to the choice of (G, A, B).

Assume that Case 2 holds. Then also by the choice of (G, A, B), $A \in \mathfrak{F}$, again a contradiction. Therefore $[A, B] = B$.

(10) *A acts on B by conjugation as a group of power automorphisms and B is an abelian group.*

By (7) $A^{\mathfrak{N}} A_p = A$. Now $A^{\mathfrak{N}}$ centralises B by Theorem 2.3.3. Also A_p normalises every subgroup of B by Theorem 2.4.2 since $A_p \cap B = 1$, A_pB is a totally permutable product of A_p and B and B is normal in A_pB. It follows that A normalises every subgroup of B.

Since $B = [A^{\mathfrak{N}} A_p, B] = [A_p, B]$, it follows that $O_q(B)$ cannot be centralised by A for each prime q dividing $|B|$. Otherwise $[A_p, B] = [A_p, O_q(B)O_{q'}(B)] = [A_p, O_{q'}(B)]$ which does not contain $O_q(B)$, a contradiction. By Lemma 1.2.9, $O_q(B)$ is abelian for all q. Hence B is a direct product of abelian subgroups and so is also abelian.

(11) *$A \cap B = A^{\mathfrak{N}} \cap B \leq Z_{\mathfrak{U}}(G)$, but $A \cap B \not\leq Z_\infty(G)$.*

Since $A \cap B$ is a p'-group by (9) and $A = A_p A^{\mathfrak{N}}$ by (7) it follows that $A \cap B = A^{\mathfrak{N}} \cap B$. Moreover, $A \cap B \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.3.5. Suppose that $A \cap B \leq Z_\infty(G)$. By (1) and (9), $B = [B, A] = [B, A_p]$ is an abelian p'-group. It also follows from (10) that $A \cap B$ is normalised by A_p. Hence by [23, A, Corollary 12.6] $A \cap B = [A \cap B, A_p]$. But $A \cap B \leq Z_\infty(G)$ implies that $A \cap B$ normalises A_p since A_p is a Sylow p-subgroup of G. Since $(A \cap B) \cap A_p = 1$ it follows that $A \cap B = [A \cap B, A_p] = 1$. Consider $G/A^{\mathfrak{N}}B$, which is a nilpotent group, and the group $G/A^{\mathfrak{N}}$. Now $G/A^{\mathfrak{N}}$ is supersoluble since it is isomorphic to a factor group of A_pB.

If Case 1 holds, then $G/B \cong A$ is an $\mathfrak{F}$-group. Using the quasi-R_0 Lemma 4.1.5, $G \in \mathfrak{F}$, contradicting the choice of (G, A, B).

Suppose Case 2 holds. Then again by the quasi-R_0 Lemma 4.1.5, it follows that $A \cong G/B \in \mathfrak{F}$, which is a contradiction.

(12) $A/A^{\mathfrak{N}}$ *is a cyclic p-group.*

Suppose that the statement is not true. Assume that there exist maximal subgroups, H and K, of A that contain $A^{\mathfrak{N}}$. Then H and K are subnormal subgroups of A and hence BH and BK are subnormal subgroups of G since B is normal in G.

Assume that Case 1 holds. Then by the minimality of (G, A, B) it follows that BH and BK belong to $\mathfrak{F}$. Therefore $G = (BH)(BK) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ which is a contradiction to the choice of (G, A, B).

Assume that Case 2 holds. Then since $G \in \mathfrak{F}$, the subnormal subgroups BL and BM also belong to $\mathfrak{F}$. By the minimality of (G, A, B), H and K belong to $\mathfrak{F}$. Therefore $A = \langle H, M\rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, which is a contradiction to the choice of (G, A, B).

The previous steps prove statements 1-5 of the lemma. What is left is to show that if $\mathfrak{U} \subseteq \mathfrak{h}$, then statement $2'$ holds. By Lemma 2.3.5, if $G = AB$ is a totally permutable product of subgroups A and B, then $A \cap B \leq Z_{\mathfrak{h}}(G)$. In this case the following holds:

(13) B *is a q-group for a prime* $q \neq p$.

Let q be a prime dividing $|B|$ and assume that the result is not true. Without loss of generality assume that q is a prime such that $A \cap O_{q'}(B) \neq 1$ by (11). From (10), G is a totally permutable product of subgroups $AO_{q'}(B)$ and $O_q(B)$, and $|G| + |AO_q(B)||O_q(B)| < |G| + |A||B|$.

Assume that Case 1 holds. Then by the minimality of (G, A, B) it follows that $AO_{q'}(B)$ belong to $\mathfrak{F}$ and $O_q(B) \in \mathfrak{F}$. Since $|G| + |AO_{q'}(B)||O_q(B)| < |G| + |A||B|$, by our choice of (G, A, B) it follows that $G \in \mathfrak{F}$, a contradiction.

Assume that Case 2 holds. By the minimality of (G, A, B), $AO_{q'}(B)$ belongs to $\mathfrak{F}$. Again by the minimality of (G, A, B) it follows that $A \in \mathfrak{F}$, which is a contradiction.

(14) B *is a cyclic q-group.*

Assume that B is not cyclic. Since B is abelian q-group and $A \cap B \neq 1$ by (11), there is a direct decomposition $B = M \times N$ where N is a cyclic group such that $A \cap N \neq 1$. So $G = (AN)M$ is a totally permutable product of subgroups AN and M by (10) and $|G| + |AN||M| < |G| + |A||B|$.
Assume that Case 1 holds. Then by the minimality of (G, A, B), it follows that $AN \in \mathfrak{F}$. Again by the minimality of (G, A, B) it follows that $G \in \mathfrak{F}$, which is a contradiction.
Assume that Case 2 holds. Then by the minimality of (G, A, B) it follows that $AN \in \mathfrak{F}$ and again by the minimality of (G, A, B) it also follows that $A \in \mathfrak{F}$, which is also a contradiction. □

As a direct consequence of Lemma 4.2.1 the following result holds.

Theorem 4.2.2. *[24, Theorem 1] Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$. Let a group $G = AB$ be a totally permutable product of subgroups A and B. Assume that $A \cap B \leq Z_\infty(G)$. Then G belongs to $\mathfrak{F}$ if and only if A and B belong to $\mathfrak{F}$.*

The next result is presented without a proof.

Theorem 4.2.3. *[24, Theorem 2] Let $\mathfrak{F}$ be a Fischer class containing $\mathfrak{U}$. Let a group $G = AB$ be a totally permutable product of subgroups A and B. Then G belongs to $\mathfrak{F}$ if and only if A and B belong to $\mathfrak{F}$.*

In [24], the authors show that if $\mathfrak{F}$ is $\mathtt{Q}$-closed and A and B are in $\mathfrak{F}$, then G is also in $\mathfrak{F}$ when G is a totally permutable product of A and B. The result is presented below with its proof.

Theorem 4.2.4. *[24, Theorem 3] Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$. Assume that whenever $G \in \mathfrak{F}$ and $N \leq Z_{\mathfrak{U}}(G)$, then $G/N \in \mathfrak{F}$(in particular, this holds for a $\mathtt{Q}$-closed Fitting class). Let a group $G = AB$ be a totally permutable product of subgroups A and B. If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.*

Proof. Assume that the theorem is not true and let $G = AB$ be a minimal counterexample as in Lemma 4.2.1 with $\mathfrak{h} = \mathfrak{E}$, the class of all finite groups.
Since B is a normal subgroup of G, A acts on B by conjugation. Let $X = [B]A$ be a semidirect product of B and A with respect to this action. By Lemma 2.0.6 there is an epimorhism $\alpha : X \to G$ such that $\mathrm{Ker}\ \alpha \cap B = 1$ given by $(b, a)\alpha = ba$ for all $a \in A, b \in B$. Now if $(b, a) \in$

Ker α, then $(b, a)\alpha = ba = 1$, which implies that $a = b^{-1}$ and consequently $a, b \in A \cap B$. Hence Ker $\alpha \leq [A \cap B](A \cap B)$, which is a direct product since $A \cap B$ is normalised by A by Lemma 4.2.1(5) and B is cyclic by Lemma 4.2.1(2′)(and hence $A \cap B$ is also normal in B).

It follows that $X/A^{\mathfrak{N}} \cong [B](A_p/(A_p \cap A^{\mathfrak{N}}))$ is a totally permutable product of abelian subgroups by Lemma 4.2.1(1) and (2′). So $X/A^{\mathfrak{N}} \in \mathfrak{U} \subseteq \mathfrak{F}$ by Theorem 3.2.3. Moreover, $X/B \cong A \in \mathfrak{F}$. On the hand $X/([B]A^{\mathfrak{N}})$ is a p-group. It follows that $X \in \mathfrak{F}$ by the quasi-R_0 Lemma 4.1.5. By Lemma 2.3.5, it follows that $A \cap B \leq Z_{\mathfrak{U}}(G)$. By Lemma 4.2.1(2′) and (5), B is cyclic and G acts on B as a group automorphisms. Hence $[A \cap B](A \cap B) \leq Z_{\mathfrak{U}}(X)$. By hypothesis $G \cong X/\text{Ker } \alpha \in \mathfrak{F}$, which is a contradiction. □

The converse of Theorem 4.2.4 was shown to hold for R_0-closed Fitting classes.

Theorem 4.2.5. *[24, Theorem 4] Let $\mathfrak{F}$ be an R_0-closed Fitting class containing $\mathfrak{U}$. Let a group $G = AB$ be a totally permutable product of subgroups A and B. If $G \in \mathfrak{F}$, then $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$.*

Proof. Assume that the theorem is not true and let $G = AB$ be a minimal counterexample as in Lemma 4.2.1 with $\mathfrak{h} = \mathfrak{E}$, the class of all finite groups. In particular, B is a nilpotent normal p'-subgroup of G and $A/A^{\mathfrak{N}}$ is a p-group for a prime p.

Since $A^{\mathfrak{N}}$ is a normal subgroup of G, A_pB acts on $A^{\mathfrak{N}}$ by conjugation. Let $C = [A^{\mathfrak{N}}]A_pB$ be a semidirect product of $A^{\mathfrak{N}}$ and A_pB with respect to this action. By Lemma 2.0.6 there is an epimorphism $\alpha : C \to G$ such that Ker $\alpha \cap A^{\mathfrak{N}} = 1$.

Firstly it is shown that C is an $\mathfrak{F}$-group. By hypothesis $C/\text{Ker } \alpha \cong G \in \mathfrak{F}$. Since A_pB is a totally permutable product of supersoluble subgroups, A_pB is supersoluble by Theorem 3.2.3. So $C/A^{\mathfrak{N}} \cong A_pB \in \mathfrak{F}$. Hence $C \cong C/(A^{\mathfrak{N}} \cap \text{ Ker } \alpha) \in \mathrm{R}_0\mathfrak{F} = \mathfrak{F}$.

We argue that $A \in \mathfrak{F}$. Since $A^{\mathfrak{N}} \cap B = 1$, it follows that $C/(A^{\mathfrak{N}} \cap B) = C \in \mathfrak{F}$. It also follows that $C/[A^{\mathfrak{N}}]B$ is a p-group and $C/A^{\mathfrak{N}} \in \mathfrak{F}$. Hence by the quasi-$\mathrm{R}_0$ Lemma 4.1.5 $C/B \in \mathfrak{F}$. It follows that $X = [A^{\mathfrak{N}}]A_p \cong C/B \in \mathfrak{F}$. By Lemma 2.0.6 there is an epimorphism $\varphi : X \to A$ such that Ker $\varphi \cap A^{\mathfrak{N}} = 1$. Now $X/A^{\mathfrak{N}}$ and $X/(A^{\mathfrak{N}} \text{ Ker } \varphi)$ are nilpotent groups. Also $X \in \mathfrak{F}$ and so by the quasi-R_0 Lemma 4.1.5 $X/ \text{ Ker } \varphi \in \mathfrak{F}$. Hence $A \in \mathfrak{F}$, our final contradiction. □

Examples of Fitting classes containing $\mathfrak{U}$ are the Fitting products $\mathfrak{F} \diamond \mathfrak{N}$ and $\mathfrak{N} \diamond \mathfrak{F}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{N}$. As mentioned in Section 4.1, $\mathfrak{F} \diamond \mathfrak{N}$ is a Fischer class and hence is covered by Theorem 4.2.3. Hauck *et al.*[24] proved that totally permutable products behaved nicely with respect to $\mathfrak{N} \diamond \mathfrak{F}$ as the following result shows.

Theorem 4.2.6. *[24, Theorem 5] Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$ of all nilpotent groups. Consider the Fitting product $\mathfrak{N} \diamond \mathfrak{F}$. Let the group $G = AB$ be a totally permutable product of subgroups A and B. Then $G \in \mathfrak{N} \diamond \mathfrak{F}$ if and only if $A \in \mathfrak{N} \diamond \mathfrak{F}$ and $B \in \mathfrak{N} \diamond \mathfrak{F}$.*

Proof. For both implications assume that B is a normal nilpotent subgroup of G and $G = AF(G)$. Also $A^{\mathfrak{N}} \cap F(A)$ is centralised by B and is a normal subgroup of A. So $A^{\mathfrak{N}} \cap F(A)$ is a normal nilpotent subgroup of G and $A^{\mathfrak{N}} \cap F(A) = A^{\mathfrak{N}} \cap F(G)$.
Assume first that $A \in \mathfrak{N} \diamond \mathfrak{F}$, that is, $A/F(A) \in \mathfrak{F}$. Since $A/A^{\mathfrak{N}}$ and $A/F(A)A^{\mathfrak{N}} \in \mathfrak{N}$ it follows that $A/(A^{\mathfrak{N}} \cap F(G)) = A/(A^{\mathfrak{N}} \cap F(A)) \in \mathfrak{F}$ by the quasi-R_0 Lemma 4.1.5. Again by the quasi-R_0 Lemma 4.1.5 applied to the group $A^{\mathfrak{N}}/(A^{\mathfrak{N}} \cap F(G)) \in \mathfrak{F}$ and the normal subgroups $(A \cap F(G))/(A^{\mathfrak{N}} \cap F(G))$ and $A^{\mathfrak{N}}/(A^{\mathfrak{N}} \cap F(G))$, it follows that $A/(F(G) \cap A) \in \mathfrak{F}$. Hence $G/F(G) = AF(G)/F(G) \cong A/(F(G) \cap A) \in \mathfrak{F}$, that is, $G \in \mathfrak{N} \diamond \mathfrak{F}$.
Conversely, assume that $G \in \mathfrak{N} \diamond \mathfrak{F}$. Note that $A^{\mathfrak{N}} \cap F(A) = A^{\mathfrak{N}} \cap F(G)$ is a normal subgroup of G and $A/(A \cap F(G)) \cong G/F(G) \in \mathfrak{F}$. Since $A/A^{\mathfrak{N}} \in \mathfrak{N}$, it follows that $A/(A^{\mathfrak{N}} \cap F(G)) = A/(A^{\mathfrak{N}} \cap F(A)) \in \mathfrak{F}$ by the quasi-R_0 Lemma 4.1.5. Applying the quasi-R_0 Lemma 4.1.5 again to $A/A^{\mathfrak{N}}F(A) \in \mathfrak{N}$, $A/(A^{\mathfrak{N}} \cap F(A)) \in \mathfrak{F}$ and $A/A^{\mathfrak{F}}$ it follows that $A/F(A) \in \mathfrak{F}$. Therefore $A \in \mathfrak{N} \diamond \mathfrak{F}$. Hence the result follows. □

Proposition 4.2.7. *[26, Theorems 3.1 and 3.2] Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$ of all supersoluble groups. Let a group $G = G_1G_2...G_n$ be a pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$.*

1. *Assume that $\mathfrak{F}$ satisfies the following property:*

If a group $G = AB$ is a totally permutable product of subgroups A and B such that if $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

If $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$, then $G \in \mathfrak{F}$.

2. *Assume that $\mathfrak{F}$ satisfies the following property:*

If a group $G = AB$ is a totally permutable product of subgroups A and B such that if $G \in \mathfrak{F}$, then $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$.

If $G \in \mathfrak{F}$, then $G_i \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$.

Proposition 4.2.8. *[25, Proposition 4] Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$ and satisfying the following condition:*

If a group $G = AB$ is the totally permutable product of subgroups A and B, then $G_{\mathfrak{F}} = A_{\mathfrak{F}}B_{\mathfrak{F}}$.

Then, for a group $G = AB$ which is the totally permutable product of subgroups A and B, the following properties hold:

1. If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.

2. If G belongs to $\mathfrak{F}$, then A and B belong to $\mathfrak{F}$.

Proof. 1. $G = AB = A_{\mathfrak{F}}B_{\mathfrak{F}} = G_{\mathfrak{F}} \in \mathfrak{F}$ and the result follows.

2. Let $G = AB$ be an $\mathfrak{F}$-group which is a totally permutable product of subgroups A and B. The proof is by induction on $|G|$. Assume that $\langle B^G \rangle = G$. Since B centralises $A^{\mathfrak{N}}$ by Theorem 2.3.3, $A^{\mathfrak{N}}$ centralises $\langle B^G \rangle = G$ since $A^{\mathfrak{N}}$ is a normal subgroup of G. Hence $A^{\mathfrak{N}} \leq Z(G)$. This implies that $A^{\mathfrak{N}} \leq Z(A)$. It follows that $A^{\mathfrak{N}} = 1$ and so $A \in \mathfrak{N} \subseteq \mathfrak{F}$. By the hypothesis, it follows that $G = G_{\mathfrak{F}} = A_{\mathfrak{F}}B_{\mathfrak{F}} = AB_{\mathfrak{F}}$ and so $B = B_{\mathfrak{F}}(A \cap B)$ by 1 since B is a totally permutable product of subgroups of $B_{\mathfrak{F}}$ and $A \cap B \in \mathfrak{F}$. So $\langle B^G \rangle$ is a proper subgroup of G. Therefore assume that both $\langle A^G \rangle$ and $\langle B^G \rangle$ are proper subgroups of G. Since $B(\langle B^G \rangle \cap A) = \langle B^G \rangle \in \mathfrak{F}$ it follows that $B \in \mathfrak{F}$ by the inductive hypothesis. Analogously, $A \in \mathfrak{F}$. □

The converse of Proposition 4.2.8 is not known to be true in general (see [25](pg. 6139)).

Proposition 4.2.9. *[25, Proposition 5] Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$ and satisfying the following condition:*

For a group $G = AB$ which is a totally permutable product of subgroups A and B, the following properties hold:

1. If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.

2. If G belongs to $\mathfrak{F}$, then A and B belong to $\mathfrak{F}$.

Suppose that the group $G = G_1G_2...G_n$ is a pairwise totally pemutable product of subgroups $G_1, G_2, ..., G_n$. Then $(G_1)_{\mathfrak{F}}(G_2)_{\mathfrak{F}}...(G_n)_{\mathfrak{F}}$ is a normal subgroup of G contained in $G_{\mathfrak{F}}$ and $(G_i)_{\mathfrak{F}} = G_i \cap G_{\mathfrak{F}}$ for all $i \in \{1, 2, ..., n\}$.

Proof. For any $i \in \{1, 2, ..., n\}$ by Lemma 2.3.5 it follows that

$$[G_i, \prod_{j=1, j\neq i}^{n} G_j] \leq Z_{\mathfrak{U}}(G) = \prod_{i=1}^{n} Z_{\mathfrak{U}}(G_i) \leq \prod_{i=1}^{n}(G_i)_{\mathfrak{F}}.$$

In particular, $\prod_{i=1}^{n}(G_i)_{\mathfrak{F}}$ is a normal subgroup of G and hence $\prod_{i=1}^{n}(G_i)_{\mathfrak{F}} \leq G_{\mathfrak{F}}$ by Statement 1 and Theorem 4.2.7. Moreover, $\prod_{i=1}^{n}(G_i)_{\mathfrak{F}} \leq \prod_{i=1}^{n}(G_i \cap G_{\mathfrak{F}})$ which is a normal subgroup of G by Lemma 2.4.4(i). It follows that $\prod_{i=1}^{n}(G_i \cap G_{\mathfrak{F}}) \in \mathfrak{F}$ and so $G_i \cap G_{\mathfrak{F}} \in \mathfrak{F}$ for all $i \in \{1, 2, ..., n\}$ by Statement 2 and Theorem 4.2.7. Consequently, $G_i \cap G_{\mathfrak{F}} = (G_i)_{\mathfrak{F}}$ for all $i \in \{1, 2, ..., n\}$ and the result follows. □

The converse of Proposition 4.2.8 holds for a Fischer class containing $\mathfrak{U}$ as the result below shows.

Theorem 4.2.10. *[25, Theorem 1] Let $\mathfrak{F}$ be a Fischer class containing $\mathfrak{U}$. Let the group $G = G_1G_2...G_n$ be a pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. Then $G_{\mathfrak{F}} = (G_1)_{\mathfrak{F}}(G_2)_{\mathfrak{F}}...(G_n)_{\mathfrak{F}}$ and $(G_i)_{\mathfrak{F}} = G_i \cap G_{\mathfrak{F}}$ for all $i \in \{1, 2, ..., n\}$.*

Remark

An example of a Fitting class satisfying the property in Proposition 4.2.8 is a Fischer class containing $\mathfrak{U}$ as Theorem 4.2.10 shows. Examples of Fitting classes satisfying properties 1 and 2 in Propositions 4.2.7 and 4.2.9 are Fitting classes in Theorems 4.2.2, 4.2.3 and 4.2.6.

4.3 Mutually Permutable Products and Fitting Classes

In this section results on mutually permutable products and Fitting classes are presented. Most results on totally permutable products have not been generalised to mutually permutable products. However, they are useful in the attempt to extend results on totally permutable products to weakly totally permutable products.
One of the questions is, for which classes of groups does Theorem 4.2.10 hold for mutually permutable products. A partial answer to this question is given below.

Theorem 4.3.1. *[7, Theorem 4.3.13] Let an SC-group $G = G_1G_2...G_n$ be a pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. Then $S(G_i) = G_i \cap S(G)$ for every $i \in \{1, 2, ..., n\}$, where $S(X)$ is the soluble radical of a group X.*

It is not known if Theorem 4.3.1 holds for non-SC groups. However, for $n = 2$, Theorem 4.3.1 can be generalised to non-SC groups as the following result shows.

Theorem 4.3.2. *[18, Theorem 4] Let a group $G = AB$ be a mutually permutable product of subgroups A and B. Then $S(A) = S(G) \cap A$ and $S(B) = S(G) \cap B$.*

Beidleman and Heineken [17] proved the following result:

Theorem 4.3.3. *[17, Theorem 1] Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups of A and B. If $G \in \mathfrak{F}$, then $A' \in \mathfrak{F}$ and $B' \in \mathfrak{F}$.*

As a consequence of Theorem 4.3.3 the following result is true:

Corollary 4.3.4. *Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups of A and B. Then $A' \cap G_{\mathfrak{F}} \leq A_{\mathfrak{F}}$ and $B' \cap G_{\mathfrak{F}} \leq B_{\mathfrak{F}}$.*

The dual statement to Theorem 4.3.3 is true as the following result shows:

Theorem 4.3.5. *[20, Corollary] Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups A and B. If $A \in \mathfrak{F}$ and $B \in \mathfrak{F}$, then $G' \in \mathfrak{F}$.*

Bochtler also showed that $\langle A_{\mathfrak{F}}, B_{\mathfrak{F}} \rangle = A_{\mathfrak{F}}B_{\mathfrak{F}}$.

Theorem 4.3.6. *[19, Theorem A] Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups A and B. Then $A_{\mathfrak{F}}B_{\mathfrak{F}}$ is a mutually permutable product of subgroups $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$.*

With the help of Theorem 4.3.6 it was shown that Corollary 4.3.4 is also dualized:

Theorem 4.3.7. *[19, Theorem B] Let $\mathfrak{F}$ be a Fitting class. Let a group $G = AB$ be a mutually permutable product of subgroups A and B. Then $G' \cap A_{\mathfrak{F}}B_{\mathfrak{F}}$ is a subnormal subgroup of G contained in $\mathfrak{F}$. In particular, $G' \cap A_{\mathfrak{F}}B_{\mathfrak{F}} \leq G_{\mathfrak{F}}$.*

4.4 Weakly Totally Permutable Products and Fitting Classes

This section is the work of the author. In this section some results on totally permutable products in the framework of Fitting classes are extended to weakly totally permutable products. In order to do this some structural results are first presented.

The first result in this section generalises Lemma 2.1.5 and extends [7, Corollary 4.2.11] to weakly totally permutable products.

Lemma 4.4.1. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B]$ is a nilpotent normal subgroup of G.*

Proof. It is clear that $[A, B]$ is a normal subgroup of G. Let $A = A_1A^{\mathfrak{U}}$ and $B = B_1B^{\mathfrak{U}}$ where A_1, B_1 is an $\mathfrak{U}$-projector of A and B, respectively. Then since B centralises $A^{\mathfrak{U}}$ and A centralises $B^{\mathfrak{U}}$ by Lemma 3.1.9, it follows that

$$[A, B] = [A_1A^{\mathfrak{U}}, B_1B^{\mathfrak{U}}] = [A_1, B_1]$$

which is contained in $(A_1B_1)'$. Since A_1B_1 is a weakly totally permutable product of supersoluble subgroups A_1 and B_1 by Theorem 3.2.2, it follows that A_1B_1 is supersoluble by Theorem 3.2.3 and hence its derived subgroup is nilpotent. The result then follows. □

The following result generalises Corollary 2.3.4 to weakly totally permutable products.

Lemma 4.4.2. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $[A, B] \leq Z_{\mathfrak{U}}(G)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Theorem 3.2.1 it follows that $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$. Let A_1 and B_1 be $\mathfrak{U}$-projectors of A and B, respectively by Theorem 1.8.5. Then $[A, B] = [A^{\mathfrak{U}}A_1, B^{\mathfrak{U}}B_1] = [A_1, B_1] \leq A_1B_1$ by Lemma 3.1.9. Moreover, A_1B_1 is a $\mathfrak{U}$-projector of G by Theorem 3.2.2. But $(A \cap B)[A, B] \leq \langle A^G \rangle \cap \langle B^G \rangle \leq C_G(A^{\mathfrak{U}}B^{\mathfrak{U}}) = C_G(G^{\mathfrak{U}})$ since $A^{\mathfrak{U}}$ and $B^{\mathfrak{U}}$ are normal subgroups of G by Lemma 2.2.2 and the fact that $A^{\mathfrak{U}}$ and $B^{\mathfrak{U}}$ centralise B and A respectively by Lemma 2.1.12. Hence $(A \cap B)[A_1, B_1] \leq C_{A_1B_1}(G^{\mathfrak{U}}) = Z_{\mathfrak{U}}(G)$ by Theorem 1.8.10. □

Corollary 4.4.3. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then*

$$G/Z_{\mathfrak{U}}(G) = (AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) \times (BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)).$$

Corollary 4.4.4. *Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $Z_{\mathfrak{U}}(A) = Z_{\mathfrak{U}}(G) \cap A$, $Z_{\mathfrak{U}}(B) = Z_{\mathfrak{U}}(G) \cap B$ and $Z_{\mathfrak{U}}(G) = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B)$. In particular, $A \cap B \leq Z_{\mathfrak{U}}(A) \cap Z_{\mathfrak{U}}(B)$.*

Proof. By Corollary 4.4.3 it follows that

$$G/Z_{\mathfrak{U}}(G) = (AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) \times (BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)).$$

Since $Z_{\mathfrak{U}}(G) \cap A$ is a supersolubly embedded normal subgroup of A, it follows that $Z_{\mathfrak{U}}(G) \cap A \leq Z_{\mathfrak{U}}(A)$. Since $Z_{\mathfrak{U}}(G/Z_{\mathfrak{U}}(G)) = 1$, it follows that $Z(AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G)) = 1$ by Corollary 4.4.3. So $A/(Z_{\mathfrak{U}}(G) \cap A)$ has a trivial $\mathfrak{U}$-hypercentre. Thus $Z_{\mathfrak{U}}(G) \cap A = Z_{\mathfrak{U}}(A)$. Analogously $Z_{\mathfrak{U}}(G) \cap B = Z_{\mathfrak{U}}(B)$.

Let $T = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B)$. Then T is a normal subgroup of G by Lemma 2.4.4(i) and $T \leq Z_{\mathfrak{U}}(G)$. Hence $T \cap A$ is a subgroup of $Z_{\mathfrak{U}}(G)$. But by the definition of T, $Z_{\mathfrak{U}}(G) \cap A$ is a subgroup of $T \cap A$. So $Z_{\mathfrak{U}}(G) \cap A = T \cap A$. On the other hand, G/T is a totally permutable product of subgroups AT/T and BT/T by Lemma 2.4.3(iv) since $A \cap B \leq T$ by Lemma 4.4.2. Hence

$$| G/T | \leq | AT/T || BT/T | = | A/(T \cap A) || B/(T \cap B) | =$$
$$| A/(Z_{\mathfrak{U}}(G) \cap A) || B/(Z_{\mathfrak{U}}(G) \cap B) | = | AZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G) || BZ_{\mathfrak{U}}(G)/Z_{\mathfrak{U}}(G) |.$$

This coincides with the order of $G/Z_{\mathfrak{U}}(G)$. Since $T \leq Z_{\mathfrak{U}}(G)$, it follows that $| G/Z_{\mathfrak{U}}(G) | \leq | G/T |$ and $| G/T | \leq | G/Z_{\mathfrak{U}}(G) |$. Consequently $T = Z_{\mathfrak{U}}(G)$ and hence the result follows. □

In an attempt to extend Theorems 4.2.3, 4.2.6 and 4.2.10, and Propositions 4.2.8 and 4.2.9, the following lemma is useful.

Lemma 4.4.5. *Let $G = AB$ be a weakly totally permutable product of subgroups A and B. Let $\mathfrak{F}$ be a Fitting class containing $\mathfrak{U}$. Suppose that one of the following cases holds:*

Case 1: *either $A, B \in \mathfrak{F}$, but $G \notin \mathfrak{F}$ with $|G| + |A| + |B|$ minimal, or*

Case 2: *$G \in \mathfrak{F}$ but not both of A and B in $\mathfrak{F}$ with $|G| + |A| + |B|$ minimal.*

Then, after interchanging the roles of A and B if necessary we have that B is a subnormal supersoluble subgroup of G, $\langle B^G \rangle$ is supersoluble and $G = A^{\mathfrak{N}} A_p \langle B^G \rangle$.

Proof. Then G has the following properties:

1. *Without loss of generality assume that B is supersoluble and that A is not supersoluble. Moreover $[A^{\mathfrak{U}}, B] = 1$.*

By Lemma 3.1.9 $[A^{\mathfrak{U}}, B] = [A, B^{\mathfrak{U}}] = 1$. Now A and B cannot both be supersoluble, because in Case 1 G would be supersoluble by Theorem 3.2.3 and in Case 2 that would contradict the choice of (G, A, B).
Suppose that neither A nor B is supersoluble. Then $A^{\mathfrak{U}} \neq 1$ and $B^{\mathfrak{U}} \neq 1$. Note that $A^{\mathfrak{U}}$ cannot be central in A and $B^{\mathfrak{U}}$ cannot be central in B otherwise either A or B would be supersoluble contradicting our supposition. Then $B \leq C_G(A^{\mathfrak{U}}) < G$ and $A \leq C_G(B^{\mathfrak{U}}) < G$. Hence $C_G(A^{\mathfrak{U}}) = B(A \cap C_G(A^{\mathfrak{U}}))$ is a weakly totally permutable product of subgroups B and

$(A \cap C_G(A^{\mathfrak{U}}))$. Assume that Case 1 holds. By the choice of (G, A, B) it follows that $C_G(A^{\mathfrak{U}}) \in \mathfrak{F}$. Analogously $C_G(B^{\mathfrak{U}}) \in \mathfrak{F}$. Since $C_G(A^{\mathfrak{U}})$ and $C_G(B^{\mathfrak{U}})$ are normal subgroups of G it follows that $G = C_G(A^{\mathfrak{U}})C_G(B^{\mathfrak{F}}) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$. Assume now that Case 2 holds. Then $C_G(A^{\mathfrak{U}})$ is an $\mathfrak{F}$-group since it is a normal subgroup of G. By the choice of (G, A, B), $B \in \mathfrak{F}$. Analogously $A \in \mathfrak{F}$ contradicting the choice of (G, A, B).

Hence assume that B is supersoluble and that A is not supersoluble.

2. *B is a subnormal subgroup of G and $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$.*

Firstly the argument is that $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$. Note first that $[A^{\mathfrak{U}}, B] = 1$ implies that $[\langle B^G \rangle, A^{\mathfrak{U}}] = 1$. Now

$$(\langle B^G \rangle \cap A)/(\langle B^G \rangle \cap A^{\mathfrak{U}}) \cong (\langle B^G \rangle \cap A)A^{\mathfrak{U}}/A^{\mathfrak{U}} \in \mathfrak{U}$$

and

$$\langle B^G \rangle \cap A^{\mathfrak{U}} \leq Z(\langle B^G \rangle \cap A^{\mathfrak{U}}),$$

so $\langle B^G \rangle \cap A$ is supersoluble. Since $A \cap B \leq \langle B^G \rangle \cap A$ it follows that $\langle B^G \rangle = B(\langle B^G \rangle \cap A)$ is a weakly totally permutable product of supersoluble subgroups and so $\langle B^G \rangle \in \mathfrak{U} \subseteq \mathfrak{F}$ by Theorem 3.2.3.

By Theorem 2.4.11 there exists L, M such that $A'(A \cap B) \leq L \leq A$ and $B'(A \cap B) \leq M \leq B$, L and M are subnormal in G and $G' \leq LM$. If $L = A$ and $M = B$, then in Case 1 it follows that $G = AB \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ and in Case 2 it follows that both A and B belong to $\mathfrak{F}$ since A and B are subnormal in G and $\mathfrak{F}$ is a Fitting class.

Suppose that $L < A$ and $M < B$.

Assume that Case 1 holds. Then BL and AM are normal $\mathfrak{F}$-subgroups of G by the choice of (G, A, B). Hence $G = (BL)(AM) \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$ which is a contradiction.

Now suppose that Case 2 holds. Then since BL and AM are normal subgroups of G, it follows that $BL, AM \in \mathfrak{F}$. By the choice of (G, A, B), $A \in \mathfrak{F}$, a contradiction. Suppose $L = A$ and $M < B$. If Case 1 holds, then $G = A\langle B^G \rangle \in \mathrm{N}_0\mathfrak{F} = \mathfrak{F}$, contradiction. If Case 2 holds, then $A \in \mathfrak{F}$, also a contradiction, since A is subnormal and $\mathfrak{F}$ is a Fitting class.

So it can be assumed that $M = B$ and $L < A$ and so B is a subnormal supersoluble subgroup of G.

3. *There exist a prime number p such that $G = A^{\mathfrak{N}} A_p \langle B^G \rangle$, where A_p is a Sylow p-subgroup of A.*

Assume that $A^{\mathfrak{N}} A_q \langle B^G \rangle$ is a proper subgroup of G for all primes q, where A_q denotes a Sylow q-subgroup of A. Then

$$A^{\mathfrak{N}} A_q \langle B^G \rangle = A^{\mathfrak{N}} A_q \langle B^G \rangle \cap AB = A^{\mathfrak{N}} A_q(\langle B^G \rangle \cap AB) = A^{\mathfrak{N}} A_q(\langle B^G \rangle \cap A)B.$$

Note also that $A^{\mathfrak{N}} A_q \langle B^G \rangle$ is a normal subgroup of G. The subgroups B and $X_q = A^{\mathfrak{N}} A_q(\langle B^G \rangle \cap A)$ and B are weakly totally permutable subgroups since $A \cap B \leq (\langle B^G \rangle \cap A) A^{\mathfrak{N}} A_q$. Assume that Case 1 holds. Then $X_q \in \mathfrak{F}$ since it is a normal subgroup of A and, by minimality of (G, A, B), $X_q B \in \mathfrak{F}$. Hence $G = \prod_{q \in \mathbb{P}} X_q B \in \mathrm{N}_0 \mathfrak{F} = \mathfrak{F}$, a contradiction. Assume Case 2 holds. Then since $X_q B$ is a normal subgroup of G it follows that $X_q B \in \mathfrak{F}$. By the choice of (G, A, B), $X_q \in \mathfrak{F}$. Hence $A = \prod_{q \in \mathbb{P}} X_q \in \mathrm{N}_0 \mathfrak{F} = \mathfrak{F}$, a final contradiction.

□

Open Question 4.4.6. *Can Theorem 4.2.3 be generalised to weakly totally permutable products?*

The following result shows that Theorem 4.2.3 can be extended to weakly totally permutable products where the Fischer class is $\mathfrak{F} \diamond \mathfrak{N}$, with $\mathfrak{F}$ a Fitting class containing $\mathfrak{U}$.

Theorem 4.4.7. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$. Consider $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{N}$ is the class of all nilpotent groups. Let $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $G \in \mathfrak{F} \diamond \mathfrak{N}$ if and only if $A \in \mathfrak{F} \diamond \mathfrak{N}$ and $B \in \mathfrak{F} \diamond \mathfrak{N}$.*

Proof. Suppose the theorem is not true and let G be a minimal counterexample. By Lemma 4.4.5, B is a subnormal supersoluble subgroup of G. So $B \leq G_{\mathfrak{F}}$ and $G = AG_{\mathfrak{F}}$ for both implications.
Assume that $A \in \mathfrak{F} \diamond \mathfrak{N}$, that is, $A/A_{\mathfrak{F}} \in \mathfrak{N}$. Then $A^{\mathfrak{N}} \leq A_{\mathfrak{F}}$ since $\mathfrak{N}$ is a formation. This

implies that $A^{\mathfrak{N}}$ is a subnormal $\mathfrak{F}$-subgroup of G by Theorem 2.4.10. So $A^{\mathfrak{N}} \leq A \cap G_{\mathfrak{F}}$. Hence $A/(A \cap G_{\mathfrak{F}}) \cong AG_{\mathfrak{F}}/G_{\mathfrak{F}} = G/G_{\mathfrak{F}} \in \mathfrak{N}$, that is, $G \in \mathfrak{F} \diamond \mathfrak{N}$, a contradiction.
Now assume that $G \in \mathfrak{F} \diamond \mathfrak{N}$. Then $G/G_{\mathfrak{F}} \cong A/(A \cap G_{\mathfrak{F}}) \in \mathfrak{N}$. So $A^{\mathfrak{N}} \leq A \cap G_{\mathfrak{F}}$, which means that $A^{\mathfrak{N}}$ is a subnormal $\mathfrak{F}$-subgroup of A. Hence $A^{\mathfrak{N}} \leq A_{\mathfrak{F}}$ and $A/A_{\mathfrak{F}} \in \mathfrak{N}$, that is, $A \in \mathfrak{F} \diamond \mathfrak{N}$ which is a contradiction. Hence the result follows. □

It is not known if Theorem 4.2.7 can be extended to weakly totally permutable products. The result below shows, however, that weakly totally permutable products behave nicely if the Fitting product $\mathfrak{N}^2 \diamond \mathfrak{F}$ is considered.

Theorem 4.4.8. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$ of all nilpotent groups. Consider the Fitting product $\mathfrak{N}^2 \diamond \mathfrak{F}$. Let the group $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$ and $B \in \mathfrak{N}^2 \diamond \mathfrak{F}$ if and only if $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$.*

Proof. Suppose the theorem is not true and let G be a minimal counterexample. Then by Lemma 4.4.5 since B is a subnormal supersoluble subgroup of G, it follows that $B \leq G_{\mathfrak{N}^2}$ and $G = AG_{\mathfrak{N}^2}$ for both implications.
Assume first that $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$, that is, $A/A_{\mathfrak{N}^2} \in \mathfrak{F}$. By the quasi-$\mathrm{R}_0$ Lemma 4.1.5 $A/(A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2}) \in \mathfrak{F}$ since $A/A^{\mathfrak{N}} \in \mathfrak{N} \subseteq \mathfrak{F}$. But $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} = A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}$ since $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} \leq A^{\mathfrak{N}} \leq A'$ is a subnormal $\mathfrak{N}^2$-subgroup of G by Theorem 2.4.10. Consider $A/(A^{\mathfrak{N}}(G_{\mathfrak{N}^2} \cap A)) \in \mathfrak{N}$ and $A/A^{\mathfrak{N}} \in \mathfrak{N} \subseteq \mathfrak{F}$. It follows that $A/(G_{\mathfrak{N}^2} \cap A) \in \mathfrak{F}$ by the quasi-R_0 Lemma 4.1.5 since $A/(A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}) \in \mathfrak{F}$. But $A/(G_{\mathfrak{N}^2} \cap A) \cong AG_{\mathfrak{N}^2}/G_{\mathfrak{N}^2} \in \mathfrak{F}$, that is $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$, a contradiction.
Assume now that $G \in \mathfrak{N}^2 \diamond \mathfrak{F}$. As in the previous argument $A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2} = A^{\mathfrak{N}} \cap G_{\mathfrak{N}^2}$ is a subnormal $\mathfrak{N}^2$-subgroup of G. Since $A/(A \cap G_{\mathfrak{N}^2}) \cong G/G_{\mathfrak{N}^2} \in \mathfrak{F}$ and $A/A^{\mathfrak{N}} \in \mathfrak{N}$, it follows that $A/(A^{\mathfrak{N}} \cap A_{\mathfrak{N}^2}) \in \mathfrak{F}$ by the quasi-R_0 Lemma 4.1.5. Applying the quasi-R_0 Lemma 4.1.5 again $A/A_{\mathfrak{N}^2} \in \mathfrak{F}$, that is, $A \in \mathfrak{N}^2 \diamond \mathfrak{F}$. Hence the result follows. □

Lemma 4.4.9. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{N}$. Let the group $G = AB$ be a weakly totally permutable product of subgroups A and B. Then $A_{\mathfrak{F}}B_{\mathfrak{F}}$ is a weakly totally permutable product of subgroups $A_{\mathfrak{F}}$ and $B_{\mathfrak{F}}$.*

Proof. By Lemma 3.1.1(ii) $A \cap B \in \mathfrak{N} \subseteq \mathfrak{F}$ is a subnormal $\mathfrak{F}$-subgroup of A and so $A \cap B \leq A_{\mathfrak{F}}$. Analogously $A \cap B \leq B_{\mathfrak{F}}$. Hence the result follows by Lemma 3.1.1(i). □

The following propositions extends Propositions 4.2.8 and 4.2.9 to weakly totally permutable products.

Proposition 4.4.10. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$ of all supersoluble groups and satisfying the following condition:*

If a group $G = AB$ is the weakly totally permutable product of subgroups A and B, then $G_{\mathfrak{F}} = A_{\mathfrak{F}}B_{\mathfrak{F}}$ *(4.1)*

Then, for a group $G = AB$ which is a weakly totally permutable product of subgroups A and B, the following properties hold:

1. If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.

2. If G belongs to $\mathfrak{F}$, then A and B belong to $\mathfrak{F}$.

Proof. 1. $G = AB = A_{\mathfrak{F}}B_{\mathfrak{F}} = G_{\mathfrak{F}} \in \mathfrak{F}$ and the result follows.

2. Let $G = AB$ be an $\mathfrak{F}$-group which is a weakly totally permutable product of subgroups A and B. The proof by induction on $|G|$. Assume that $\langle B^G \rangle = G$. Since B centralises $A^{\mathfrak{U}}$ by Lemma 3.1.9, $A^{\mathfrak{U}}$ centralises $\langle B^G \rangle = G$ and so $A^{\mathfrak{U}} \leq Z(G)$. This implies that $A^{\mathfrak{U}} \leq Z(A)$. It follows that $A^{\mathfrak{U}} = 1$ and so $A \in \mathfrak{U} \subseteq \mathfrak{F}$. By the hypothesis, it follows that $G = G_{\mathfrak{F}} = A_{\mathfrak{F}}B_{\mathfrak{F}} = AB_{\mathfrak{F}}$ and so $B = B_{\mathfrak{F}}(A \cap B) \in \mathfrak{F}$ by 1 since B is a totally permutable product of subgroups of $B_{\mathfrak{F}}$ and $A \cap B \in \mathfrak{F}$. Therefore assume that both $\langle A^G \rangle$ and $\langle B^G \rangle$ are proper subgroups of G. Since $B(\langle B^G \rangle \cap A) = \langle B^G \rangle \in \mathfrak{F}$ it follows that $B \in \mathfrak{F}$ by the inductive hypothesis. Analogously, $A \in \mathfrak{F}$. □

Proposition 4.4.11. *Let $\mathfrak{F}$ be a Fitting class containing the class $\mathfrak{U}$ of all supersoluble groups and satisfying the following condition:*

For a group $G = AB$ which is a weakly totally permutable product of subgroups A and B, the following properties hold:

(a) If A and B belong to $\mathfrak{F}$, then G belongs to $\mathfrak{F}$.

(b) If G belongs to $\mathfrak{F}$, then A and B belong to $\mathfrak{F}$. *(4.2)*

Then $A_{\mathfrak{F}}B_{\mathfrak{F}}$ is a normal subgroup of G, $A_{\mathfrak{F}}B_{\mathfrak{F}} \leq G_{\mathfrak{F}}$ and, $A_{\mathfrak{F}} = A \cap G_{\mathfrak{F}}$ and $B_{\mathfrak{F}} = B \cap G_{\mathfrak{F}}$

Proof. By Theorem 1.7.7 it follows $Z_{\mathfrak{U}}(A), Z_{\mathfrak{U}}(B) \in \mathfrak{U} \subseteq \mathfrak{F}$. By Lemma 4.4.2 it also follows that $[A, B] \leq Z_{\mathfrak{U}}(G) = Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B) \leq A_{\mathfrak{F}}B_{\mathfrak{F}}$. In particular, $A_{\mathfrak{F}}B_{\mathfrak{F}}$ is a normal subgroup of G. By (a) $A_{\mathfrak{F}}B_{\mathfrak{F}} \leq G_{\mathfrak{F}}$. Moreover, $A_{\mathfrak{F}}B_{\mathfrak{F}} \leq (A \cap G_{\mathfrak{F}})(B \cap G_{\mathfrak{F}}) \trianglelefteq G_{\mathfrak{F}}$ by Lemma 2.4.4(i). So $(A \cap G_{\mathfrak{F}})(B \cap G_{\mathfrak{F}}) \in \mathfrak{F}$ and $A \cap G_{\mathfrak{F}} \in \mathfrak{F}$, $B \cap G_{\mathfrak{F}} \in \mathfrak{F}$ by (b). Hence $A \cap G_{\mathfrak{F}} = A_{\mathfrak{F}}$ and $B \cap G_{\mathfrak{F}} = B_{\mathfrak{F}}$, and the result follows. □

Remark

The Fitting classes satisfying properties *(4.1)* and *(4.2)* in Propositions 4.4.10 and 4.4.11 respectively, are $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$ (Theorem 4.4.7) and $\mathfrak{N}^2 \diamond \mathfrak{F}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{N}$(Theorem 4.4.8).
It remains to be determined if Theorem 4.2.10 can be extended to weakly totally permutable products where the Fischer class is $\mathfrak{F} \diamond \mathfrak{N}$, where $\mathfrak{F}$ is a Fitting class containing $\mathfrak{U}$.

In this chapter it has been shown that some results on weakly totally can be extended to weakly totally permutable products in the framework of Fitting classes.

Chapter 5

Products of Finite Groups and Group Classes

The main aim of the thesis was achieved since some results on totally permutable products were extended to weakly totally permutable products. In Chapter 2 the following results were presented.

Theorem 2.2.4 *Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If for all $i \in \{1, 2, ..., n\}$ the subgroups G_i are in $\mathfrak{F}$, then $G \in \mathfrak{F}$.*

Theorem 2.3.1 *Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$ such that either $\mathfrak{F}$ is saturated or $\mathfrak{F}$ is a formation of soluble groups. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. Then $G^{\mathfrak{F}} = G_1^{\mathfrak{F}}G_2^{\mathfrak{F}}...G_n^{\mathfrak{F}}$.*

Theorem 2.3.2 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Consider a group $G = G_1G_2...G_n$ which is the pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$. If A_i is an $\mathfrak{F}$-projector of G_i for all $i \in \{1, 2, ..., n\}$, then the product $A_1A_2...A_n$ is an $\mathfrak{F}$-projector of G.*

Theorem 2.5.4 *Let the group $G = G_1G_2...G_n$ be the pairwise totally permutable product of* *subgroups $G_1, G_2, ..., G_n$. Then G is an SC-group if and only if G_i is an SC-group for all $i \in \{1, 2, ..., n\}$.*

In Chapter 3 the results above were generalised when $n = 2$ as the results below show.

Theorem 3.2.1 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$.*

Theorem 3.2.2 *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A_1 and B_1 are $\mathfrak{F}$-projectors of A and B respectively, then A_1B_1 is an $\mathfrak{F}$-projector of G.*

Theorem 3.2.3 *Let $\mathfrak{F}$ be a formation containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. If A and B belong to $\mathfrak{F}$, then G also belongs to $\mathfrak{F}$.*

Theorem 3.3.1 *Let the group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then G is an SC-group if and only if A and B are SC-groups.*

In this chapter, the thesis is closed by looking at ideas on some of the open questions highlighted in Chapter 3 and Chapter 4.

In Chapter 2 the following result was presented:

Theorem 2.3.3 *Let a group $G = AB$ be the totally permutable product of subgroups A and B. Then $[A^{\mathfrak{N}}, B] = [A, B^{\mathfrak{N}}] = 1$.*

In particular $A^{\mathfrak{N}}$ and $B^{\mathfrak{N}}$ are normal subgroups of G in the case above. There is Open Question 3.1.11 in Chapter 3 which asks:

If $G = AB$ is a weakly totally permutable product of subgroups A and B, does $A^{\mathfrak{N}}$ centralise B?

In this chapter we attempt to answer this question:

If $G = AB$ is a weakly totally permutable product of subgroups A and B, are $A^{\mathfrak{N}}$ and $B^{\mathfrak{N}}$ normal subgroups of G?

Considered is a counterexample below.

Lemma 5.0.12. *Let a group $G = AB$ be a weakly totally permutable product of subgroups A and B. Consider the following property : The subgroup $A^{\mathfrak{N}}$ is a normal subgroup of G.*
Suppose the property does not hold for weakly totally permutable products in general and let $G = AB$ be a counterexample with $|G| + |A| + |B|$ minimal.
Then G is supersoluble.

Proof. Consider $x \in B$ and let $D = A \cap B$. If $D\langle x\rangle < B$, then
$|A\langle x\rangle| + |A| + |D\langle x\rangle| < |G| + |A| + |B|$. Moreover, $A\langle x\rangle$ is a weakly totally permutable product of subgroups A and $D\langle x\rangle$ by Lemma 3.1.1(i). By the choice of (G, A, B), it follows that $A^{\mathfrak{N}}$ is normalised by $D\langle x\rangle$. If for all $x \in B$, $D\langle x\rangle < B$, then $A^{\mathfrak{N}}$ is normalised by $B = \langle D\langle x\rangle \mid x \in B\rangle$, a contradiction to the choice of (G, A, B). Hence $B = D\langle x\rangle$ for some $x \in B$. Since $B = D\langle x\rangle$ is a totally permutable product of subgroups D and $\langle x\rangle$, it follows that B is supersoluble by Theorem 3.2.3.
Using Lemma 2.4.8, let N be a normal subgroup of G such that $N \leq A$ or $N \leq B$. Then G/N is a weakly totally permutable product of AN/N and BN/N by Lemma 3.1.3. By the choice of (G, A, B), it follows that $BN/N \leq N_{G/N}(A^{\mathfrak{N}}N/N)$, that is, $A^{\mathfrak{N}}N$ is a normal subgroup of G. Suppose that $A^{\mathfrak{U}} \neq 1$. Then $A^{\mathfrak{U}}$ is a normal subgroup of G by Lemma 3.1.10 and hence $A^{\mathfrak{N}}A^{\mathfrak{U}} = A^{\mathfrak{N}}$ is a normal subgroup of G, a contradiction. Hence $A^{\mathfrak{U}} = 1$ and A is supersoluble. By Theorem 3.2.3, G is supersoluble and the result follows. □

Most results on totally permutable products presented in Chapter 2 were extended to weakly totally permutable products when $n = 2$.
Considering Open Question 3.2.4, the key question is:

If a group $G = G_1G_2...G_n$ is a pairwise totally permutable product of subgroups $G_1, G_2, ..., G_n$, is G supersoluble ? (5.1)

First let us look at PST-groups. Recall that a subgroup A of G is S-permutable in G if A permutes with every Sylow subgroup of G (see [7], Definition 1.2.1, pg. 10).

Definition 5.0.13. *A group G is a PST-group if S-permutability is a transitive in G, that is, if A is an S-permutable subgroup of B and B is an S-permutable subgroup of G, then A is an S-permutable subgroup of G.*

The following result shows how a PST-group is characterised.

Lemma 5.0.14. *[31, Lemma 2.3] A group G is a PST-group if and only if every subnormal subgroup of G is an S-permutable subgroup of G*

Ballester-Bolinches *et al.* [6] proved that if the factors are PST-groups, then the product is an SC-group as the following result shows.

Theorem 5.0.15. *[6, Theorem 6] Let a group $G = G_1G_2...G_n$ the pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. If G_i is a PST-group for all $i \in \{1, 2, ..., n\}$. Then G is an SC-group.*

Using Theorem 2.4.6 it follows that:

Corollary 5.0.16. *Let a group $G = G_1G_2...G_n$ be the pairwise mutually permutable product of subgroups $G_1, G_2, ..., G_n$. If G_i is a soluble PST-group for all $i \in \{1, 2, ..., n\}$, then G is supersoluble.*

For a pairwise weakly totally permutable product in an attempt to answer question *(5.1)*, a minimal counterexample is considered below.

Lemma 5.0.17. *Let a group $G = G_1G_2...G_n$ be the pairwise weakly totally permutable product of supersoluble subgroups $G_1, G_2, ..., G_n$. Suppose that*
G is not supersoluble and let G be a counterexample with $|G| + |G_1| + |G_2| + ... + |G_n|$ minimal. Then

(i) there exists G_i for some $i \in \{1, 2, ..., n\}$ such that $G_i^{\mathfrak{N}} \neq 1$,

(ii) if p is the largest prime dividing $|G|$, then the Sylow p-subgroup P of G is normal in G,

(iii) either G/P is supersoluble or $P = \prod_{i \neq j, i=1, j=1}^{n} (G_i \cap G_j)_p$, where $(G_i \cap G_j)_p$ is the Sylow p-subgroup of $G_i \cap G_j$.

Proof. (i) If G_i is nilpotent for all $i \in \{1, 2, ..., n\}$, then by Corollary 5.0.16, G is supersoluble, a contradiction to the choice of $(G, G_1, G_2, ..., G_n)$. Hence (i) follows.

(ii) By Corollary 2.4.6 G is soluble. Let P_i be the Sylow p-subgroup of G_i, where p is the largest prime dividing $|G|$. Then $P = P_1 P_2 ... P_n$ is a Sylow p-subgroup of G. Let Q_i be a Hall p'-subgroup of G_i. Then $Q = Q_1 Q_2 ... Q_n$ is a Hall p'-subgroup of G.
Then P_i is normalised by Q_i for all $i \in \{1, 2, ..., n\}$ since G_i is supersoluble. Moreover $P_i(G_i \cap G_j)Q_j$ is a weakly totally permutable product of subgroups $P_i(G_i \cap G_j)$ and $(G_i \cap G_j)Q_j$ by Lemma 3.1.1(i). Also $P_i(G_i \cap G_j)$ and $(G_i \cap G_j)Q_j$ are supersoluble by Theorem 3.2.3. Hence $P_i(G_i \cap G_j)Q_j$ is supersoluble for all $i, j \in \{1, 2, ..., n\}$, $i \neq j$. Therefore P_i is normalised by Q_j, that is, P_i is normalised by $Q = \prod_{i=1}^{n} Q_i$, the Hall p'-subgroup of G. It follows that P is normalised by Q and P is the normal Sylow p-subgroup of G. Hence (ii) follows.

(iii) Suppose that there exists $i \in \{1, 2, ..., n\}$ such that $T_i = \prod_{i \neq j, j=1}^{n} (G_i \cap G_j)_p < P_i$. Without loss of generality assume that $T_1 < P_1$. Then $M = (T_1 Q_1) G_2 G_3 ... G_n$ is a pairwise weakly totally permutable product of supersoluble subgroups $T_1 Q_1, G_2, G_3, ..., G_n$. Moreover, $|M| + |T_1 Q_1| + |G_2| + |G_3| + ... + |G_n < |G| + |G_1| + ... + |G_n|$. Hence by the choice of $(G, G_1, G_2, ..., G_n)$, it follows that M is supersoluble. Therefore G/P, which is isomorphic to a factor group of M, is supersoluble.
Otherwise $T_i = P_i$ and $\prod_{i=1}^{n} (\prod_{i \neq j, j=1}^{n} (G_i \cap G_j)_p)$. Hence (iii) follows. □

On Fitting classes one of the unanswered questions is:

Is there a Fitting class satisfying property (4.1) of Theorem 4.4.10?

By considering $\mathfrak{H} \diamond \mathfrak{N}$, where $\mathfrak{H}$ is a Fitting class containing $\mathfrak{U}$, an attempt is made to answer this question.

Lemma 5.0.18. *Let $\mathfrak{F} = \mathfrak{H} \diamond \mathfrak{N}$, where $\mathfrak{H}$ is a Fitting class containing $\mathfrak{U}$. Let a group $G = AB$ be the weakly totally permutable product of subgroups A and B. Then $A_{\mathfrak{F}} = A \cap G_{\mathfrak{F}}$, $B_{\mathfrak{F}} = B \cap G_{\mathfrak{F}}$, and $A = A_{\mathfrak{F}}\langle a\rangle$ and $B = B_{\mathfrak{F}}\langle b\rangle$ for some $a \in A$ and $b \in B$.*

Proof. The proof is by induction on $|G| + |A| + |B|$. Now G has the following properties:

(1) *Let $L = A_{\mathfrak{F}}B_{\mathfrak{F}}$. Then $A_{\mathfrak{F}} = A \cap G_{\mathfrak{F}}$ and $B_{\mathfrak{F}} = B \cap G_{\mathfrak{F}}$, and L is a normal subgroup of G.*

This is by Theorem 4.4.7 and Proposition 4.4.11.

(2) $[G, G_{\mathfrak{F}}] \leq L$*; in particular,* $G_{\mathfrak{F}}/L$ *is abelian.*

By Lemma 4.4.2

$$[A, B] \leq Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{U}}(A)Z_{\mathfrak{U}}(B) \leq L.$$

Hence

$$[AL, G] = [A, G][L, G] \leq [A, A][A, B][L, G] \leq AL$$

and

$$[BL, G] = [B, G][L, G] \leq [B, B][A, B][L, G] \leq BL$$

and so $H = AL$ and $K = BL$ are normalised by G and $G = HK$.
If either $H = G$ or $K = G$, then

$$G_{\mathfrak{F}} = A_{\mathfrak{F}}(A \cap G_{\mathfrak{F}})B_{\mathfrak{F}}(B \cap G_{\mathfrak{F}}) = A_{\mathfrak{F}}B_{\mathfrak{F}}$$

and the result follows. Therefore assume that H and K are normal subgroups of G and $G = HK$. So $H_{\mathfrak{F}}$ and $K_{\mathfrak{F}}$ are subgroups of $G_{\mathfrak{F}}$. Since $H_{\mathfrak{F}} = G_{\mathfrak{F}} \cap AL = (G_{\mathfrak{F}} \cap A)L = L$ and $K_{\mathfrak{F}} = G_{\mathfrak{F}} \cap BL = (B_{\mathfrak{F}} \cap A)L = L$ by (1), it follows that

$$[G, G_{\mathfrak{F}}] = [HK, G_{\mathfrak{F}}] = [H, G_{\mathfrak{F}}][K, G_{\mathfrak{F}}] \leq H_{\mathfrak{F}} K_{\mathfrak{F}} = L.$$

If L is a proper subgroup of $G_{\mathfrak{F}}$, since $G_{\mathfrak{F}}/L$ is abelian by (2), consider a normal subgroup $T = L\langle x \rangle$ of $G_{\mathfrak{F}}$ such that $x^p \in L$, where x is a p-element and $x = ab$ for some $a \in A$ and $b \in B$, where a and b are p-element and p is a prime number p.

(3) $A = A_{\mathfrak{F}}\langle a \rangle$ *and* $B = B_{\mathfrak{F}}\langle b \rangle$.

Let $S = A_{\mathfrak{F}}\langle a \rangle B$. It follows that $T \leq S_{\mathfrak{F}}$ since T is a subnormal subgroup of G, $T \leq S$ and $T \in \mathfrak{F}$. Let $U = A_{\mathfrak{F}}\langle a \rangle \leq A$ and $V = B_{\mathfrak{F}}\langle b \rangle \leq B$. So $S = A_{\mathfrak{F}}\langle a \rangle B$ is a weakly totally permutable product of subgroups $A_{\mathfrak{F}}\langle a \rangle$ and B by Lemma 4.4.9 and Lemma 3.1.1.
Now if U is a proper subgroup of A, then $|S| + |A_{\mathfrak{F}}\langle a \rangle| + |B| < |G| + |A| + |B|$ and by the inductive hypothesis, it follows that $S_{\mathfrak{F}} = U_{\mathfrak{F}} B_{\mathfrak{F}}$. Then

$$x \in S_{\mathfrak{F}} \cap G_{\mathfrak{F}} = (U_{\mathfrak{F}} \cap G_{\mathfrak{F}})B_{\mathfrak{F}} \leq (A \cap G_{\mathfrak{F}})B_{\mathfrak{F}} = A_{\mathfrak{F}} B_{\mathfrak{F}} = L$$

using (1). Then $x \in L$, contradicting the choice of x. Therefore $U = A$.
Analogously $V = B$ and the hence the result follows. □

Lemmas 5.0.12 and 5.0.17 may be used for further study on weakly totally permutable products.

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